1 Proof Theory

1.1 Inheriting Natural Deduction of Propositional Logic

If we consider propositions as nullary predicates, propositional logic is a sub-language of predicate logic. It will come as no surprise that we can translate the rules for natural deduction in propositional logic directly to predicate logic. Each of the following rules applies to any formulas $\phi$ and $\psi$ of predicate logic.

\[
\begin{align*}
\phi & \quad \psi \\
\hline
\phi \land \psi & \quad [\land i] \\
\phi \land \psi & \quad [\land e_1] \\
\phi \land \psi & \quad [\land e_2]
\end{align*}
\]

\[
\begin{align*}
\phi & \\
\hline
\phi \lor \psi & \quad [\lor i_1] \\
\phi \lor \psi & \quad [\lor i_2] \\
\phi \lor \psi & \quad [\lor e]
\end{align*}
\]

\[
\begin{align*}
\phi \lor \psi & \\
\hline
\phi & \quad \psi \\
\hline
\phi & \quad \ldots \\
\hline
\chi & \quad \psi \\
\hline
\chi & \\
\hline
\chi & \quad \ldots \\
\hline
\chi & \quad [\lor e]
\end{align*}
\]
1.2 Equality

We mentioned in “Semantics of Predicate Logic” that equality is usually interpreted to mean identity, which means that in a model $a =^M b$ holds if and only if $a$ and $b$ are the same elements of the model’s universe. It is safe to assume $t = t$ for any term $t$, because both sides of the equation will evaluate to the same element, regardless of the context (environment) in which we operate. The following equality introduction rule expresses this reasoning.

$$
[= i] \\
t = t
$$

The next rule, equality elimination, allows us to replace a term $t_1$ by another term $t_2$, provided that $t_1 = t_2$ is already proven. More precisely, in order to prove a formula $\psi$, in which a term $t_2$ appears (possibly multiple times), it is sufficient to prove $t_1 = t_2$ and the formula $\psi'$ that results from $\psi$ by replacing $t_2$ by $t_1$. The rule stated below uses a formula $\phi$ in which a free variable $x$ represents the positions of $t_2$ in $\psi$, thus $\psi = [x \Rightarrow t_2]\phi$, and $\psi' = [x \Rightarrow t_1]\phi$. 

$$
[\rightarrow i] \\
f \rightarrow \psi \quad \phi \rightarrow \psi \\
\psi

[\rightarrow e] \\
f \rightarrow \psi \quad \phi \rightarrow \psi \\
\psi
$$

$$
[\neg i] \\
f \quad \neg f \\
\neg f

[\neg e] \\
f \quad \neg f \\
\neg f
$$

$$
[\bot e] \\
\bot \quad \neg \neg f \\
\phi

[\neg \neg e] \\
\bot \quad \neg \neg f \\
\phi
$$
\[
\begin{align*}
t_1 &= t_2 & [x \Rightarrow t_1] \phi \\
\hline
[x \Rightarrow t_2] \phi
\end{align*}
\]

Using these two rules, we show: We show:

\[f(z) = g(z) \vdash h(g(z)) = h(f(z))\]
as follows:

<table>
<thead>
<tr>
<th>Line</th>
<th>Statement</th>
<th>Justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( f(z) = g(z) )</td>
<td>premise</td>
</tr>
<tr>
<td>2</td>
<td>( h(f(z)) = h(f(z)) )</td>
<td>= i</td>
</tr>
<tr>
<td>3</td>
<td>( h(g(z)) = h(f(z)) )</td>
<td>= e 1,2</td>
</tr>
</tbody>
</table>

Note that the formula \( h(g(z)) = h(f(z)) \) in Line 3 has the form \( [x \Rightarrow t_2] \phi \), where \( t_2 \) is \( g(z) \) and \( \phi \) is \( h(z) = h(f(z)) \). If we use \( f(z) \) for \( t_1 \), then Rule = e asks us to prove \( t_1 = t_2 \) (Line 1), and \( [x \Rightarrow t_1] \phi \) (Line 2).

1.3 Universal Quantification

Elimination of Universal Quantification Once you have proven \( \forall x \phi \), you can replace \( x \) by any term \( t \) in \( \phi \), provided that \( t \) is free for \( x \) in \( \phi \), and thus “eliminate” the universal quantification.

\[
\begin{align*}
\forall x \phi \\
\hline
[x \Rightarrow t] \phi
\end{align*}
\]

This rule is justified by the semantics of \( \forall x \phi \), since in a particular context (environment) any term \( t \) denotes a value in the model, and \( \phi \) holds for all such values, if \( \forall x \phi \) holds in the model.

In \( t \) any function symbols of the logic, as well as variables that are known in the context can be used.

Example 1. We shall prove: \( S(g(john)) \), \( \forall x(S(x) \rightarrow \neg L(x)) \vdash \neg L(g(john)) \)

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<td>( \forall x \ e \ 2 )</td>
</tr>
<tr>
<td>4</td>
<td>( \neg L(g(john)) )</td>
<td>( \rightarrow e \ 3,1 )</td>
</tr>
</tbody>
</table>

Introduction of Universal Quantification The introduction rule for universal quantification is more complicated. Let us consider a new kind of box that allows us to introduce a fresh variable. For example,
is a box in which the variable $z$ can be used in terms, as in

\[
\begin{array}{c}
  : \\
  f(z) = f(z) \\
  : 
\end{array}
\]

Let us say we introduce a variable $x_0$ in a box. Without any assumptions on $x_0$, we prove a formula $\psi$, in which $x_0$ appears. The fact that $x_0$ appears in $\psi$, we can characterize by writing $\psi$ as $[x \Rightarrow x_0] \phi$. Since we have not made any assumptions on $x_0$ within the box, we have shown that $[x \Rightarrow x_0] \phi$ holds for all possible instantiations of $x$ by values of the universe; in other words, we can conclude $\forall x \phi$.

\[
\begin{array}{c}
  : \\
  [x \Rightarrow x_0] \phi \\
  \hline
  \forall x \phi
\end{array}
\]

The variable $x_0$ must be fresh; we cannot introduce the same variable twice in nested boxes. Freshness of course guarantees that $x_0$ is free for $x$ in $\phi$.

**Example 2.** We shall prove the sequent $\forall x (P(x) \rightarrow Q(x)), \forall x P(x) \vdash \forall x Q(x)$.

1. $\forall x (P(x) \rightarrow Q(x))$ premise
2. $\forall x P(x)$ premise
3. $P(x_0)$ premise $\forall x \; e \; 1$ $x_0$
4. $P(x_0)$ $\forall x \; e \; 2$
5. $Q(x_0)$ $\rightarrow e \; 3,4$
6. $\forall x Q(x)$ $\forall x \; i \; 3-5$

**1.4 Existential Quantification**

**Introduction of Existential Quantification** For existential quantification, the easy direction is introduction.

\[
[x \Rightarrow t] \phi \quad \underline{[\exists x \; i]} \quad \exists x \phi
\]

In order to prove $\exists x \phi$, it suffices to find a term $t$ as “witness”, provided—as usual—that $t$ is free for $x$ in $\phi$. 

4
Example 3. Assume that the set $F$ contains a nullary function symbol $c$, and that the set $P$ contains a unary predicate symbol $P$. We should be able to prove:

$$\forall x P(x) \vdash \exists x P(x)$$

since at least the constant $c$ should have the property $P$, once we know that all elements of the universe has the property $P$. The corresponding proof follows:

1. $\forall x P(x)$ premise
2. $[x \Rightarrow c] P(x)$ $\forall x \ e \ 1$
3. $\exists x P(x)$ $\exists x \ i \ 2$

Elimination of Existential Quantification Finally, for elimination of existential quantification, we combine the two kinds of boxes; we simultaneously introduce a fresh variable and an assumption.

$$\exists x \phi \quad \left[ x \Rightarrow x_0 \phi \right] \quad \vdash \quad \chi$$

If we know $\exists x \phi$, we know that there exist at least one object $x$ for which $\phi$ holds. We call that element $x_0$, and assume $[x \Rightarrow x_0] \phi$ within a box in which we introduce $x_0$. Without assumptions on $x_0$, we prove a formula $\chi$, in which $x_0$ does not appear. Since we have not made any assumptions on $x_0$, we can conclude from $\exists x \phi$ that $\chi$ holds.

Example 4. We prove the following sequent:

$$\forall x (P(x) \rightarrow Q(x)), \exists x P(x) \vdash \exists x Q(x)$$

1. $\forall x (P(x) \rightarrow Q(x))$ premise
2. $\exists x P(x)$ premise
3. $P(x_0)$ assumption $x_0$
4. $P(x_0) \rightarrow Q(x_0)$ $\forall x \ e \ 1$
5. $Q(x_0)$ $\rightarrow e \ 4,3$
6. $\exists x Q(x)$ $\exists x \ i \ 5$
7. $\exists x Q(x)$ $\exists x \ e \ 2,3-6$

Note that $\exists x Q(x)$ within the box does not contain $x_0$, and therefore can be “exported” from the box.

Example 5. We prove the following sequent:

$$\forall x (Q(x) \rightarrow R(x)), \exists x (P(x) \land Q(x)) \vdash \exists x (P(x) \land R(x))$$
\[\forall x (Q(x) \rightarrow R(x))\] premise

1. \[\exists x (P(x) \land Q(x))\] premise

2. \[P(x_0) \land Q(x_0)\] assumption \(x_0\)

3. \[Q(x_0) \rightarrow R(x_0)\] \(\forall x \in 1\)

4. \[Q(x_0)\] \(\land \epsilon_2 \ 3\)

5. \[R(x_0)\] \(\rightarrow \epsilon \ 4, 5\)

6. \[P(x_0)\] \(\land \epsilon_1 \ 3\)

7. \[P(x_0) \land R(x_0)\] \(\land \epsilon \ 7, 6\)

8. \[\exists x (P(x) \land R(x))\] \(\exists x \in 8\)

9. \[\exists x (P(x) \land R(x))\] \(\exists x \in 2, 3-9\)

Note that variables introduced by a box must be fresh! The following is not a proof, since the variable \(x_0\) is introduced in nested boxes.

1. \[\exists x P(x)\] premise

2. \[\forall x (P(x) \rightarrow Q(x))\] premise

3. \[P(x_0)\] assumption \(x_0\)

4. \[P(x_0) \rightarrow Q(x_0)\] \(\forall x \in 2\)

5. \[Q(x_0)\] \(\rightarrow \epsilon \ 5, 4\)

6. \[Q(x_0)\] \(\exists x \in 1, 4-6\)

7. \[\forall y Q(y)\] \(\forall y \in 3-7\)

1.5 Equivalences

We write \(\phi \equiv \psi\) iff \(\phi \vdash \psi\) and also \(\psi \vdash \phi\).

Lemma 1.

\[-\forall x \phi \equiv \exists x \neg \phi\]

\[-\exists x \phi \equiv \forall x \neg \phi\]

\[\forall x \forall y \phi \equiv \forall y \forall x \phi\]

\[\exists x \exists y \phi \equiv \exists y \exists x \phi\]

\[\forall x \phi \land \forall x \psi \equiv \forall x (\phi \land \psi)\]

\[\exists x \phi \lor \exists x \psi \equiv \exists x (\phi \lor \psi)\]

Proof. We shall prove the left-to-right directions of the first and fourth statement, and leave the remaining proofs to the reader. The proves are actually schemas; actual sequents and proofs are obtained by replacing \(\phi\) and \(\psi\) with arbitrary formulas in a particular predicate logic.

- \(-\forall x \phi \vdash \exists x \neg \phi\)
\[ \neg \forall x \phi \quad \text{premise} \]

\[ \neg \exists x \neg \phi \quad \text{assumption} \]

\[ x_0 \]

\[ \neg [x \Rightarrow x_0]\phi \quad \text{assumption} \]

\[ \exists x \neg \phi \quad \exists x \ i \ 4 \]

\[ \bot \quad \neg e \ 5, \ 2 \]

\[ [x \Rightarrow x_0]\phi \quad \text{PBC} \ 4-6 \]

\[ \forall x \phi \quad \forall x \ i \ 3-7 \]

\[ \bot \quad \neg e \ 8, \ 1 \]

\[ \exists x \neg \phi \quad \text{PBC} \ 2-9 \]

- \( \exists x \exists y \phi \vdash \exists y \exists x \phi \)

If \( x \) and \( y \) are the same variable, the left and write hand side are the same formula, and thus the sequent holds through a simple argument (for example conjunction introduction followed by elimination).

Assume now that \( x \) and \( y \) are different variables.

\[ \exists x \exists y \phi \quad \text{premise} \]

\[ [x \Rightarrow x_0](\exists y \phi) \quad \text{assumption} \]

\[ \exists y([x \Rightarrow x_0]\phi) \quad \text{def of subst (} x, \ y \text{ different)} \]

\[ [y \Rightarrow y_0][x \Rightarrow x_0]\phi \quad \text{assumption} \]

\[ [x \Rightarrow x_0][y \Rightarrow y_0]\phi \quad \text{def of subst (} x, \ y, \ x_0, \ y_0 \text{ different)} \]

\[ \exists x[y \Rightarrow y_0]\phi \quad \exists x \ i \ 5 \]

\[ \exists y\exists x\phi \quad \exists y \ i \ 6 \]

\[ \exists y\exists x\phi \quad \exists y \ e \ 3, \ 4-7 \]

\[ \exists y\exists x\phi \quad \exists x \ e \ 1, \ 2-8 \]

\[ \square \]

**Exercise 1.** Prove the remaining directions of the statements in Lemma 1.

**Lemma 2.** Assuming that \( x \) is not free in \( \psi \), the following sequents hold:

\[
\begin{align*}
\forall x \phi \land \psi & \vdash \forall x(\phi \land \psi) \\
\forall x \phi \lor \psi & \vdash \forall x(\phi \lor \psi) \\
\exists x \phi \land \psi & \vdash \exists x(\phi \land \psi) \\
\exists x \phi \lor \psi & \vdash \exists x(\phi \lor \psi)
\end{align*}
\]

**Exercise 2.** Prove the statements of Lemma 2.

2 Soundness, Completeness, Compactness

The following result justifies the use of natural deduction in predicate logic.
Theorem 1 (Soundness and Completeness of Predicate Logic).

\[ \phi_1, \ldots, \phi_n \models \psi \iff \phi_1, \ldots, \phi_n \vdash \psi \]

The theorem states that every valid sequent can be proven, and every sequent that can be proven is valid. This theorem was proven by Kurt Gödel in 1929 in his doctoral dissertation. A description of his proof, as well as the proofs of the following theorems, is beyond the scope of this chapter.

Theorem 2. The decision problem of validity in predicate logic is undecidable: no program exists which, given any language in predicate logic and any formula \( \phi \) in that language, decides whether \( \models \phi \).

Proof. (sketch)

- Establish that the Post Correspondence Problem (PCP) is undecidable
- Translate an arbitrary PCP, say \( C \), to a formula \( \phi \).
- Establish that \( \models \phi \) holds if and only if \( C \) has a solution.
- Conclude that validity of predicate logic formulas is undecidable.

Theorem 3. Let \( \Gamma \) be a (possibly infinite) set of sentences of predicate logic. If all finite subsets of \( \Gamma \) are satisfiable, the \( \Gamma \) itself is satisfiable.

Theorem 4 (Löwenheim-Skolem Theorem). Let \( \psi \) be a sentence of predicate logic such that for any natural number \( n \geq 1 \) there is a model of \( \psi \) with at least \( n \) elements. Then \( \psi \) has a model with infinitely many elements.