09—Modal Logic II

CS 3234: Logic and Formal Systems

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1. Review of Modal Logic

2. Some Modal Logics

3. Natural Deduction in Modal Logic

4. Knowledge in Multi-Agent Systems

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation
Syntax and Semantics
Valid Formulas wrt Modalities
Correspondence Theory

1 Review of Modal Logic
- Motivation
- Syntax and Semantics
- Valid Formulas wrt Modalities
- Correspondence Theory

2 Some Modal Logics

3 Natural Deduction in Modal Logic

4 Knowledge in Multi-Agent Systems

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation
Syntax and Semantics
Valid Formulas wrt Modalities
Correspondence Theory

Notions of Truth

- Often, it is not enough to distinguish between "true" and "false".
- We need to consider *modalities* if truth, such as:
  - necessity
  - time
  - knowledge by an agent
- Modal logic constructs a framework using which modalities can be formalized and reasoning methods can be established.

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation
Syntax and Semantics
Valid Formulas wrt Modalities
Correspondence Theory

Syntax of Basic Modal Logic

$$\phi \quad ::= \quad \top \mid \bot \mid p \mid (\neg\phi) \mid (\phi \wedge \phi)$$
$$\mid (\phi \vee \phi) \mid (\phi \to \phi)$$
$$\mid (\phi \leftrightarrow \phi)$$
$$\mid (\Box\phi) \mid (\Diamond\phi)$$

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation
Syntax and Semantics
Valid Formulas wrt Modalities
Correspondence Theory

Kripke Models

Definition

A model $\mathcal{M}$ of propositional modal logic over a set of propositional atoms A is specified by three things:

1. A W of *worlds*;

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation
Syntax and Semantics
Valid Formulas wrt Modalities
Correspondence Theory

Kripke Models

Definition

A model $\mathcal{M}$ of propositional modal logic over a set of propositional atoms A is specified by three things:

1. A W of *worlds*;
2. a relation R on W, meaning $R \subseteq W \times W$, called the *accessibility relation*;

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation
Syntax and Semantics
Valid Formulas wrt Modalities
Correspondence Theory

Kripke Models

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A model $\mathcal{M}$ of propositional modal logic over a set of propositional atoms A is specified by three things:

1. A W of *worlds*;

2. a relation R on W, meaning $R \subseteq W \times W$, called the *accessibility relation*;

3. a function $L : W \to A \to \{T, F\}$, called *labeling function*.

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation
Syntax and Semantics
Valid Formulas wrt Modalities
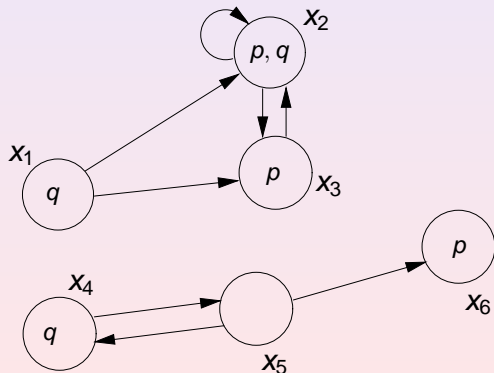Correspondence Theory

Example

$$W = \{x_1, x_2, x_3, x_4, x_5, x_6\}$$

$$R = \{(x_1, x_2), (x_1, x_3), (x_2, x_2), (x_2, x_3), (x_3, x_2), (x_4, x_5), (x_5, x_4), (x_5, x_6)\}$$

$$L = \{(x_1, \{q\}), (x_2, \{p, q\}), (x_3, \{p\}), (x_4, \{q\}), (x_5, \{\}), (x_6, \{p\})\}$$

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation
Syntax and Semantics
Valid Formulas wrt Modalities
Correspondence Theory

When is a formula true in a possible world?

Definition

Let $\mathcal{M} = (W, R, L)$, $x \in W$, and ϕ a formula in basic modal logic. We define $x \Vdash \phi$ via structural induction:

- $x \Vdash \top$

- $x \nVdash \bot$

- $x \Vdash p$ iff $p \in L(x)(p) = T$

- $x \Vdash \neg\phi$ iff $x \nVdash \phi$

- $x \Vdash \phi \wedge \psi$ iff $x \Vdash \phi$ and $x \Vdash \psi$

- $x \Vdash \phi \vee \psi$ iff $x \Vdash \phi$ or $x \Vdash \psi$
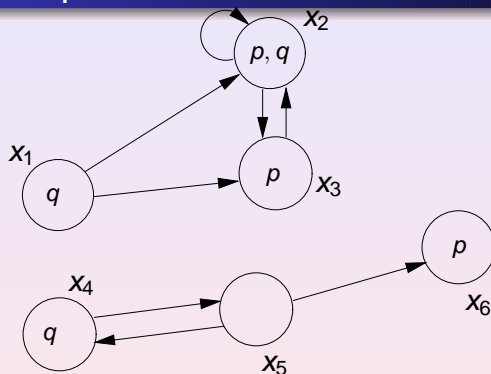
- ...

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation
Syntax and Semantics
Valid Formulas wrt Modalities
Correspondence Theory

When is a formula true in a possible world?

Definition (continued)

Let $\mathcal{M} = (W, R, L)$, $x \in W$, and ϕ a formula in basic modal logic. We define $x \Vdash \phi$ via structural induction:

- ...
- $x \Vdash \phi \rightarrow \psi$ iff $x \Vdash \psi$, whenever $x \Vdash \phi$
- $x \Vdash \phi \leftrightarrow \psi$ iff ($x \Vdash \phi$ iff $x \Vdash \psi$)
- $x \Vdash \Box \phi$ iff for each $y \in W$ with $R(x, y)$, we have $y \Vdash \phi$
- $x \Vdash \Diamond \phi$ iff there is a $y \in W$ such that $R(x, y)$ and $y \Vdash \phi$.

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation
Syntax and Semantics
Valid Formulas wrt Modalities
Correspondence Theory

Example

- $x_1 \Vdash q$

- $x_1 \Vdash \Diamond q$, $x_1 \nVdash \Box q$

- $x_5 \nVdash \Box p$, $x_5 \nVdash \Box q$, $x_5 \nVdash \Box p \vee \Box q$, $x_5 \Vdash \Box(p \vee q)$

- $x_6 \Vdash \Box \phi$ holds for all ϕ, but $x_6 \nVdash \Diamond \phi$

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation
Syntax and Semantics
Valid Formulas wrt Modalities
Correspondence Theory

A Range of Modalities

In a particular context $\Box\phi$ could mean:

- It is necessarily true that ϕ
- It will always be true that ϕ
- It ought to be that ϕ
- Agent Q believes that ϕ
- Agent Q knows that ϕ
- After any execution of program P, ϕ holds.

Since $\Diamond\phi \equiv \neg\Box\neg\phi$, we can infer the meaning of $\Diamond$ in each context.

A Range of Modalities

From the meaning of $\Box\phi$, we can conclude the meaning of $\Diamond\phi$, since $\Diamond\phi \equiv \neg\Box\neg\phi$:

$\Box\phi$	$\Diamond\phi$
It is necessarily true that ϕ	It is possibly true that ϕ
It will always be true that ϕ	Sometime in the future ϕ
It ought to be that ϕ	It is permitted to be that ϕ
Agent Q believes that ϕ	ϕ is consistent with Q's beliefs
Agent Q knows that ϕ	For all Q knows, ϕ
After any run of P, ϕ holds.	After some run of P, ϕ holds

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation
Syntax and Semantics
Valid Formulas wrt Modalities
Correspondence Theory

Formula Schemes that hold wrt some Modalities

$\Box\phi$	$\Box\phi \to \phi$	$\Box\phi \to \Box\Box\phi$	$\Diamond\phi \to \Box\Diamond\phi$	$\Diamond\top$	$\Box\phi \to \Diamond\phi$	$\Box\phi \lor \Box\neg\phi$	$\Box(\phi \to \psi) \land \Box\phi \to \Box\psi$	$\Diamond\phi \land \Diamond\psi \to \Diamond(\phi \land \psi)$
It is necessary that ϕ	√	√	√	√	√	×	√	×
It will always be that ϕ	×	√	×	×	×	×	√	×
It ought to be that ϕ	×	×	×	√	×	×	×	×
Agent Q believes that ϕ	×	√	√	√	√	×	√	×
Agent Q knows that ϕ	√	√	√	√	√	×	√	×
After running P, ϕ	×	×	×	×	×	×	√	×

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation
Syntax and Semantics
Valid Formulas wrt Modalities
Correspondence Theory

Modalities lead to Interpretations of R

$\Box\phi$	$R(x,y)$
It is necessarily true that ϕ	y is possible world according to info at x
It will always be true that ϕ	y is a future world of x
It ought to be that ϕ	y is an acceptable world according to the information at x
Agent Q believes that ϕ	y could be the actual world according to Q's beliefs at x
Agent Q knows that ϕ	y could be the actual world according to Q's knowledge at x
After any execution of P, ϕ holds	y is a possible resulting state after execution of P at x

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation
Syntax and Semantics
Valid Formulas wrt Modalities
Correspondence Theory

Possible Properties of R

- reflexive: for every $w \in W$, we have $R(x, x)$.
- symmetric: for every $x, y \in W$, we have $R(x, y)$ implies $R(y, x)$.
- serial: for every x there is a y such that $R(x, y)$.
- transitive: for every $x, y, z \in W$, we have $R(x, y)$ and $R(y, z)$ imply $R(x, z)$.
- Euclidean: for every $x, y, z \in W$ with $R(x, y)$ and $R(x, z)$, we have $R(y, z)$.
- functional: for each x there is a unique y such that $R(x, y)$.
- linear: for every $x, y, z \in W$ with $R(x, y)$ and $R(x, z)$, we have $R(y, z)$ or $y = z$ or $R(z, y)$.
- total: for every $x, y \in W$, we have $R(x, y)$ and $R(y, x)$.
- equivalence: reflexive, symmetric and transitive.

Reflexivity and Transitivity

Theorem

The following statements are equivalent:

- R is reflexive;
- $\mathcal{F}$ satisfies $\Box\phi \rightarrow \phi$;
- $\mathcal{F}$ satisfies $\Box p \rightarrow p$;

Theorem

The following statements are equivalent:

- R is transitive;
- $\mathcal{F}$ satisfies $\Box\phi \rightarrow \Box\Box\phi$;
- $\mathcal{F}$ satisfies $\Box p \rightarrow \Box\Box p$;

Formula Schemes and Properties of R

name	formula scheme	property of R
T	$\Box\phi \rightarrow \phi$	reflexive
B	$\phi \rightarrow \Box\Diamond\phi$	symmetric
D	$\Box\phi \rightarrow \Diamond\phi$	serial
4	$\Box\phi \rightarrow \Box\Box\phi$	transitive
5	$\Diamond\phi \rightarrow \Box\Diamond\phi$	Euclidean
	$\Box\phi \leftrightarrow \Diamond\phi$	functional
	$\Box(\phi \wedge \Box\phi \rightarrow \psi) \vee \Box(\psi \wedge \Box\psi \rightarrow \phi)$	linear

1 Review of Modal Logic

2 Some Modal Logics
- K
- KT45
- KT4

3 Natural Deduction in Modal Logic

4 Knowledge in Multi-Agent Systems

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

Which Formula Schemes to Choose?

Definition

Let $\mathcal{L}$ be a set of formula schemes and $\Gamma \cup \{\psi\}$ a set of formulas of basic modal logic.

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

Which Formula Schemes to Choose?

Definition

Let $\mathcal{L}$ be a set of formula schemes and $\Gamma \cup \{\psi\}$ a set of formulas of basic modal logic.

- A set of formula schemes is said to be *closed* iff it contains all substitution instances of its elements.

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

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- A set of formula schemes is said to be *closed* iff it contains all substitution instances of its elements.
- Let $\mathcal{L}_c$ be the smallest closed superset of $\mathcal{L}$.

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

Which Formula Schemes to Choose?

Definition

Let $\mathcal{L}$ be a set of formula schemes and $\Gamma \cup \{\psi\}$ a set of formulas of basic modal logic.

- A set of formula schemes is said to be *closed* iff it contains all substitution instances of its elements.
- Let $\mathcal{L}_c$ be the smallest closed superset of $\mathcal{L}$.
- Γ entails ψ in $\mathcal{L}$ iff $\Gamma \cup \mathcal{L}_c$ semantically entails ψ. We say $\Gamma \models_{\mathcal{L}} \psi$.

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

Examples of Modal Logics: K

K is the weakest modal logic, $\mathcal{L} = \emptyset$.

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

Examples of Modal Logics: KT45

$\mathcal{L} = \{T, 4, 5\}$

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

Examples of Modal Logics: KT45

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Used for reasoning about knowledge.

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

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Used for reasoning about knowledge.

name	formula scheme	property of R
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Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

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- T: Truth: agent Q only knows true things.

- 4: Positive introspection: If Q knows something, he knows that he knows it.

- 5: Negative introspection: If Q doesn't know something, he knows that he doesn't know it.

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

Explanation of Negative Introspection

name	formula scheme	property of R
...	...	...
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Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

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$$\Diamond\phi \quad \rightarrow \quad \Box\Diamond\phi$$

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

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5	$\Diamond\phi \rightarrow \Box\Diamond\phi$	Euclidean

$$\Diamond\phi \quad \rightarrow \quad \Box\Diamond\phi$$
$$\Diamond\neg\psi \quad \rightarrow \quad \Box\Diamond\neg\psi$$

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

Explanation of Negative Introspection

name	formula scheme	property of R
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5	$\Diamond\phi \rightarrow \Box\Diamond\phi$	Euclidean

$$\Diamond\phi \quad \rightarrow \quad \Box\Diamond\phi$$

$$\Diamond\neg\psi \quad \rightarrow \quad \Box\Diamond\neg\psi$$

$$\neg\Box\neg\neg\psi \quad \rightarrow \quad \Box\neg\Box\neg\neg\psi$$

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

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$$\neg\Box\neg\neg\psi \quad \rightarrow \quad \Box\neg\Box\neg\neg\psi$$

$$\neg\Box\psi \quad \rightarrow \quad \Box\neg\Box\psi$$

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

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$$\Diamond\phi \rightarrow \Box\Diamond\phi$$
$$\Diamond\neg\psi \rightarrow \Box\Diamond\neg\psi$$
$$\neg\Box\neg\neg\psi \rightarrow \Box\neg\Box\neg\neg\psi$$
$$\neg\Box\psi \rightarrow \Box\neg\Box\psi$$

If Q doesn't know ψ, he knows that he doesn't know ψ.

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

Correspondence for KT45

Accessibility relations for KT45

KT45 hold if and only if R is reflexive (T), transitive (4) and Euclidean (5).

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

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Fact on such relations

A relation is reflexive, transitive and Euclidean iff it is reflexive, transitive and symmetric, i.e. iff it is an equivalence relation.

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

Collapsing Modalities

Theorem

Any sequence of modal operators and negations is KT45 is equivalent to one of the following: $-, \Box, \Diamond, \neg, \neg\Box$, and $\neg\Diamond$, where $-$ indicates the absence of any negation or modality.

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

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Examples

- $\Box\Box\phi \equiv \Box\phi$

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

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Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

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- $\Diamond\Box\phi \equiv \Diamond\phi$
- $\neg\Diamond\neg\phi \equiv \Box\phi$

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

Examples of Modal Logics: KT4

$\mathcal{L} = \{T, 4\}$

Used for partial evaluation in computer science.

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

Examples of Modal Logics: KT4

$\mathcal{L} = \{T, 4\}$

Used for partial evaluation in computer science.

name	formula scheme	property of R
T	$\Box \phi \to \phi$	reflexive
4	$\Box \phi \to \Box\Box\phi$	transitive

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

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Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

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Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

Correspondence for KT4

Accessibility relations for KT4

KT4 hold if and only if R is reflexive (T), and transitive (4).

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

Correspondence for KT4

Accessibility relations for KT4

KT4 hold if and only if R is reflexive (T), and transitive (4).

Definition

A reflexive and transitive relation is called a *preorder*.

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

Collapsing Modalities

Theorem

Any sequence of modal operators and negations is KT4 is equivalent to one of the following:

$-, \Box, \Diamond, \Box\Diamond, \Diamond\Box, \Box\Diamond\Box, \Diamond\Box\Diamond, \neg, \neg\Box, \neg\Diamond, \neg\Box\Diamond, \neg\Diamond\Box, \neg\Box\Diamond\Box$, and $\neg\Diamond\Box\Diamond$.

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

Connection to Intuitionistic Logic

Definition

A model of intuitionistic propositional logic is a model
$\mathcal{M} = (W, R, L)$ of KT4 such that $R(x, y)$ always implies
$L(x)(p) \rightarrow L(y)(p)$.

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

Satisfaction in Intuitionistic Logic

Definition

We change the definition of $x \Vdash \phi$ as follows:

- $x \Vdash \top$
- $x \not\Vdash \bot$
- $x \Vdash p$ iff $p \in L(x)$
- $x \Vdash \phi \wedge \psi$ iff $x \Vdash \phi$ and $x \Vdash \psi$
- $x \Vdash \phi \vee \psi$ iff $x \Vdash \phi$ or $x \Vdash \psi$

as usual,

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

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- $x \Vdash \phi \vee \psi$ iff $x \Vdash \phi$ or $x \Vdash \psi$

as usual, but now:

- $x \Vdash \neg\phi$ **iff** for all y with $R(x, y)$, we have $y \nVdash \phi$

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

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as usual, but now:

- $x \Vdash \neg\phi$ **iff** for all y with $R(x, y)$, we have $y \nVdash \phi$
- $x \Vdash \phi \rightarrow \psi$ **iff** for all y with $R(x, y)$, we have $y \Vdash \psi$ whenever we have $y \Vdash \phi$.

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

Example

Let $W = \{x, y\}$, $R = \{(x, x), (x, y), (y, y)\}$,
$L(x)(p) = F, L(y)(p) = T$.

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

Example

Let $W = \{x, y\}$, $R = \{(x, x), (x, y), (y, y)\}$,
$L(x)(p) = F$, $L(y)(p) = T$. Does $x \Vdash p \vee \neg p$ hold?

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

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Since

- $x \Vdash \phi \vee \psi$ iff $x \Vdash \phi$ or $x \Vdash \psi$

we would need: $x \Vdash \neg p$.

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

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- $x \Vdash \neg \phi$ **iff** for all y with $R(x, y)$, we have $y \not\Vdash \phi$

we cannot establish $x \Vdash \neg p$.

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

K
KT45
KT4

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Since

- $x \Vdash \neg \phi$ **iff** for all y with $R(x, y)$, we have $y \not\Vdash \phi$

we cannot establish $x \Vdash \neg p$.

Idea

Do not allow "assumptions", even if they exhaust all
possibilities.

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

More Boxes
Rules
Extra Rules
Example

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

More Boxes
Rules
Extra Rules
Example

Dashed Boxes

Idea

In addition to proof boxes for assumptions, we introduce *blue boxes* that express knowledge about an *arbitrary accessible world*.

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

More Boxes
Rules
Extra Rules
Example

Dashed Boxes

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In addition to proof boxes for assumptions, we introduce *blue boxes* that express knowledge about an *arbitrary accessible world*.

Rules about blue boxes

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

More Boxes
Rules
Extra Rules
Example

Dashed Boxes

Idea

In addition to proof boxes for assumptions, we introduce *blue boxes* that express knowledge about an *arbitrary accessible world*.

Rules about blue boxes

- Whenever $\Box\phi$ occurs in a proof, ϕ may be put into a subsequent blue box.

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

More Boxes
Rules
Extra Rules
Example

Dashed Boxes

Idea

In addition to proof boxes for assumptions, we introduce *blue boxes* that express knowledge about an *arbitrary accessible world*.

Rules about blue boxes

- Whenever $\Box\phi$ occurs in a proof, ϕ may be put into a subsequent blue box.
- Whenever ϕ occurs at the end of a blue box, $\Box\phi$ may be put after that blue box.

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

More Boxes
Rules
Extra Rules
Example

Rules for □

Introduction of □:

$$\frac{\boxed{\begin{array}{c} \vdots \\ \phi \end{array}}}{\Box \phi} [\Box \ i]$$

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

More Boxes
Rules
Extra Rules
Example

Rules for □

Elimination of □:

$$\frac{\Box\phi}{\begin{array}{|c|}\hline \\ \vdots \\ \phi \\ \vdots \\ \\ \hline\end{array}} [\Box\ e]$$

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

More Boxes
Rules
Extra Rules
Example

Extra Rules for KT45

$$\frac{\Box\phi}{\phi}\,[T] \qquad \frac{\Box\phi}{\Box\Box\phi}\,[4] \qquad \frac{\neg\Box\phi}{\Box\neg\Box\phi}\,[5]$$

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

More Boxes
Rules
Extra Rules
Example
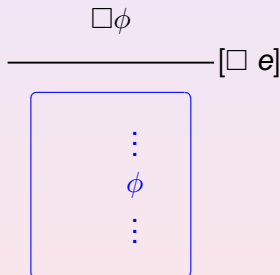
Example Proof

$$\vdash_K \Box p \wedge \Box q \rightarrow \Box(p \wedge q)$$

1	$\Box p \wedge \Box q$	assumption
2	$\Box p$	$\wedge e_1$ 1
3	$\Box q$	$\wedge e_2$ 1
4	p	$\Box e$ 2
5	q	$\Box e$ 3
6	$p \wedge q$	$\wedge i$ 4,5
7	$\Box(p \wedge q)$	$\Box i$ 4–6
8	$\Box p \wedge \Box q \rightarrow \Box(p \wedge q)$	$\rightarrow i$ 1–7

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation: The Wise Women Puzzle
Modal Logic KT45^n
Models of KT45^n
Formulation of Wise-Women Puzzle

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation: The Wise Women Puzzle
Modal Logic KT45^n
Models of KT45^n
Formulation of Wise-Women Puzzle

Wise Women Puzzle

- Three wise women, each wearing one hat, among three available red hats and *two* available white hats

Review of Modal Logic	**Motivation: The Wise Women Puzzle**
Some Modal Logics	Modal Logic KT45^n
Natural Deduction in Modal Logic	Models of KT45^n
Knowledge in Multi-Agent Systems	Formulation of Wise-Women Puzzle

Wise Women Puzzle

- Three wise women, each wearing one hat, among three available red hats and *two* available white hats
- Each wise woman is wise, can see other hats but not her own

Review of Modal Logic	**Motivation: The Wise Women Puzzle**
Some Modal Logics	Modal Logic KT45^n
Natural Deduction in Modal Logic	Models of KT45^n
Knowledge in Multi-Agent Systems	Formulation of Wise-Women Puzzle

Wise Women Puzzle

- Three wise women, each wearing one hat, among three available red hats and *two* available white hats
- Each wise woman is wise, can see other hats but not her own
- Queen asks first wise woman: Do you know the color of your hat.

Review of Modal Logic	**Motivation: The Wise Women Puzzle**
Some Modal Logics	Modal Logic KT45^n
Natural Deduction in Modal Logic	Models of KT45^n
Knowledge in Multi-Agent Systems	Formulation of Wise-Women Puzzle

Wise Women Puzzle

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- Each wise woman is wise, can see other hats but not her own
- Queen asks first wise woman: Do you know the color of your hat.
  Answer: No

Review of Modal Logic   **Motivation: The Wise Women Puzzle**
Some Modal Logics   Modal Logic KT45^n
Natural Deduction in Modal Logic   Models of KT45^n
Knowledge in Multi-Agent Systems   Formulation of Wise-Women Puzzle

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- Three wise women, each wearing one hat, among three available red hats and *two* available white hats
- Each wise woman is wise, can see other hats but not her own
- Queen asks first wise woman: Do you know the color of your hat.
  Answer: No
- Queen asks second wise woman: Do you know the color of your hat.

Review of Modal Logic	**Motivation: The Wise Women Puzzle**
Some Modal Logics	Modal Logic KT45^n
Natural Deduction in Modal Logic	Models of KT45^n
Knowledge in Multi-Agent Systems	Formulation of Wise-Women Puzzle

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  Answer: No
- Queen asks second wise woman: Do you know the color of your hat.
  Answer: No

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation: The Wise Women Puzzle
Modal Logic KT45^n
Models of KT45^n
Formulation of Wise-Women Puzzle

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- Queen asks first wise woman: Do you know the color of your hat.
  Answer: No
- Queen asks second wise woman: Do you know the color of your hat.
  Answer: No
- Queen asks third wise woman: Do you know the color of your hat?

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation: The Wise Women Puzzle
Modal Logic KT45^n
Models of KT45^n
Formulation of Wise-Women Puzzle

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- Queen asks first wise woman: Do you know the color of your hat.
  Answer: No
- Queen asks second wise woman: Do you know the color of your hat.
  Answer: No
- Queen asks third wise woman: Do you know the color of your hat?
- What is her answer?

Review of Modal Logic	**Motivation: The Wise Women Puzzle**
Some Modal Logics	Modal Logic KT45^n
Natural Deduction in Modal Logic	Models of KT45^n
Knowledge in Multi-Agent Systems	Formulation of Wise-Women Puzzle

Motivation

Reasoning about knowledge

We saw that KT45 can be used to reason about an agent's knowledge.

Review of Modal Logic	**Motivation: The Wise Women Puzzle**
Some Modal Logics	Modal Logic KT45^n
Natural Deduction in Modal Logic	Models of KT45^n
Knowledge in Multi-Agent Systems	Formulation of Wise-Women Puzzle

Motivation

Reasoning about knowledge

We saw that KT45 can be used to reason about an agent's knowledge.

Difficulty

We have three agents (queen does not count), not just one. We want them to be able to reason about *each others* knowledge.

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation: The Wise Women Puzzle
Modal Logic KT45^n
Models of KT45^n
Formulation of Wise-Women Puzzle

Motivation

Reasoning about knowledge

We saw that KT45 can be used to reason about an agent's knowledge.

Difficulty

We have three agents (queen does not count), not just one. We want them to be able to reason about *each others* knowledge.

Idea

Introduce a □ operator for each agent, and a □ operator for a group of agents.

Review of Modal Logic	Motivation: The Wise Women Puzzle
Some Modal Logics	**Modal Logic KT45^n**
Natural Deduction in Modal Logic	Models of KT45^n
Knowledge in Multi-Agent Systems	Formulation of Wise-Women Puzzle

Modal Logic KT45^n

Agents

Assume a set $\mathcal{A} = \{1, 2, \ldots, n\}$ of agents.

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation: The Wise Women Puzzle
Modal Logic KT45^n
Models of KT45^n
Formulation of Wise-Women Puzzle

Modal Logic KT45^n

Agents

Assume a set $\mathcal{A} = \{1, 2, \ldots, n\}$ of agents.

Modal connectives

Replace $\Box$ by:

- K_i for each agent i
- E_G for any subset G of $\mathcal{A}$

Review of Modal Logic	Motivation: The Wise Women Puzzle
Some Modal Logics	**Modal Logic KT45^n**
Natural Deduction in Modal Logic	Models of KT45^n
Knowledge in Multi-Agent Systems	Formulation of Wise-Women Puzzle

Modal Logic KT45^n

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Assume a set $\mathcal{A} = \{1, 2, \ldots, n\}$ of agents.

Modal connectives

Replace $\Box$ by:

- K_i for each agent i
- E_G for any subset G of $\mathcal{A}$

Example

$K_1\ p \wedge K_1 \neg K_2\ K_1\ p$ means:

> *Agent 1 knows p, and also that Agent 2 does not know that Agent 1 knows p.*

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation: The Wise Women Puzzle
Modal Logic KT45^n
Models of KT45^n
Formulation of Wise-Women Puzzle

Common Knowledge

"Everyone knows that everyone knows"

In KT45^n, $E_G E_G \phi$ is stronger than $E_G \phi$.

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation: The Wise Women Puzzle
Modal Logic KT45^n
Models of KT45^n
Formulation of Wise-Women Puzzle

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In KT45^n, $E_G\, E_G\, \phi$ is stronger than $E_G\, \phi$.

"Everyone knows everyone knows everyone knows"

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Review of Modal Logic	Motivation: The Wise Women Puzzle
Some Modal Logics	**Modal Logic KT45^n**
Natural Deduction in Modal Logic	Models of KT45^n
Knowledge in Multi-Agent Systems	Formulation of Wise-Women Puzzle

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In KT45^n, $E_G\ E_G\ E_G\ \phi$ is stronger than $E_G\ E_G\ \phi$.

Common knowledge

The infinite conjunction $E_G\phi \wedge E_G\ E_G\phi \wedge \ldots$ is called "common knowledge of ϕ", denoted, $C_G\ \phi$.

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation: The Wise Women Puzzle
Modal Logic KT45^n
Models of KT45^n
Formulation of Wise-Women Puzzle

Distributed Knowledge

Combine knowledge

If intelligent agents communicate with each other and use the knowledge each have, they can discover new knowledge.

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation: The Wise Women Puzzle
Modal Logic KT45^n
Models of KT45^n
Formulation of Wise-Women Puzzle

Distributed Knowledge

Combine knowledge

If intelligent agents communicate with each other and use the knowledge each have, they can discover new knowledge.

Distributed knowledge

The operator $D_G\phi$ is called "distributed knowledge of ϕ", denoted, $D_G \, \phi$.

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation: The Wise Women Puzzle
Modal Logic KT45^n
Models of KT45^n
Formulation of Wise-Women Puzzle

Models of KT45^n

Definition

A model $\mathcal{M} = (W, (R_i)_{i \in \mathcal{A}}, L)$ of the multi-modal logic KT45^n is specified by three things:

Review of Modal Logic	Motivation: The Wise Women Puzzle
Some Modal Logics	Modal Logic KT45^n
Natural Deduction in Modal Logic	**Models of KT45^n**
Knowledge in Multi-Agent Systems	Formulation of Wise-Women Puzzle

Models of KT45^n

Definition

A model $\mathcal{M} = (W, (R_i)_{i \in \mathcal{A}}, L)$ of the multi-modal logic KT45^n is specified by three things:

1. A set W, whose elements are called *worlds*;

Review of Modal Logic	Motivation: The Wise Women Puzzle
Some Modal Logics	Modal Logic KT45^n
Natural Deduction in Modal Logic	**Models of KT45^n**
Knowledge in Multi-Agent Systems	Formulation of Wise-Women Puzzle

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A model $\mathcal{M} = (W, (R_i)_{i \in \mathcal{A}}, L)$ of the multi-modal logic KT45^n is specified by three things:

1. A set W, whose elements are called *worlds*;

2. For each $i \in \mathcal{A}$ a relation R_i on W, meaning $R_i \subseteq W \times W$, called the accessibility relations;

Review of Modal Logic	Motivation: The Wise Women Puzzle
Some Modal Logics	Modal Logic KT45^n
Natural Deduction in Modal Logic	**Models of KT45^n**
Knowledge in Multi-Agent Systems	Formulation of Wise-Women Puzzle

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2. For each $i \in \mathcal{A}$ a relation R_i on W, meaning $R_i \subseteq W \times W$, called the accessibility relations;

3. A labeling function $L : W \to \mathcal{P}(\texttt{Atoms})$.

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation: The Wise Women Puzzle
Modal Logic KT45^n
Models of KT45^n
Formulation of Wise-Women Puzzle

Semantics of KT45^n

Definition

Take a model $\mathcal{M} = (W, (R_i)_{i \in \mathcal{A}}, L)$ and a world $x \in W$. We define $x \Vdash \phi$ via structural induction:

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation: The Wise Women Puzzle
Modal Logic KT45^n
Models of KT45^n
Formulation of Wise-Women Puzzle

Semantics of KT45^n

Definition

Take a model $\mathcal{M} = (W, (R_i)_{i \in \mathcal{A}}, L)$ and a world $x \in W$. We define $x \Vdash \phi$ via structural induction:

- $x \Vdash p$ iff $p \in L(x)$

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation: The Wise Women Puzzle
Modal Logic KT45^n
Models of KT45^n
Formulation of Wise-Women Puzzle

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- $x \Vdash \neg\phi$ iff $x \nVdash \phi$

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation: The Wise Women Puzzle
Modal Logic KT45^n
Models of KT45^n
Formulation of Wise-Women Puzzle

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- $x \Vdash p$ iff $p \in L(x)$
- $x \Vdash \neg\phi$ iff $x \nVdash \phi$
- $x \Vdash \phi \wedge \psi$ iff $x \Vdash \phi$ and $x \Vdash \psi$

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation: The Wise Women Puzzle
Modal Logic KT45^n
Models of KT45^n
Formulation of Wise-Women Puzzle

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- $x \Vdash \neg\phi$ iff $x \not\Vdash \phi$
- $x \Vdash \phi \wedge \psi$ iff $x \Vdash \phi$ and $x \Vdash \psi$
- $x \Vdash \phi \vee \psi$ iff $x \Vdash \phi$ or $x \Vdash \psi$

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation: The Wise Women Puzzle
Modal Logic KT45^n
Models of KT45^n
Formulation of Wise-Women Puzzle

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- $x \Vdash \phi \wedge \psi$ iff $x \Vdash \phi$ and $x \Vdash \psi$

- $x \Vdash \phi \vee \psi$ iff $x \Vdash \phi$ or $x \Vdash \psi$

- $x \Vdash \phi \rightarrow \psi$ iff $x \Vdash \psi$, whenever $x \Vdash \phi$

- ...

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation: The Wise Women Puzzle
Modal Logic KT45^n
Models of KT45^n
Formulation of Wise-Women Puzzle

Semantics of KT45^n (continued)

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Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation: The Wise Women Puzzle
Modal Logic KT45^n
Models of KT45^n
Formulation of Wise-Women Puzzle

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- ...
- $x \Vdash K_i \phi$ iff for each $y \in W$ with $R_i(x, y)$, we have $y \Vdash \phi$

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation: The Wise Women Puzzle
Modal Logic KT45^n
Models of KT45^n
Formulation of Wise-Women Puzzle

Semantics of KT45^n (continued)

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- ...
- $x \Vdash K_i \phi$ iff for each $y \in W$ with $R_i(x, y)$, we have $y \Vdash \phi$
- $x \Vdash E_G \phi$ iff for each $i \in G$, $x \Vdash K_i \phi$.

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation: The Wise Women Puzzle
Modal Logic KT45^n
Models of KT45^n
Formulation of Wise-Women Puzzle

Semantics of KT45^n (continued)

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- ...
- $x \Vdash K_i \phi$ iff for each $y \in W$ with $R_i(x, y)$, we have $y \Vdash \phi$
- $x \Vdash E_G \phi$ iff for each $i \in G$, $x \Vdash K_i \phi$.
- $x \Vdash C_G \phi$ iff for each $k \geq 1$, we have $x \Vdash E_G{}^k \phi$.

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation: The Wise Women Puzzle
Modal Logic KT45^n
Models of KT45^n
Formulation of Wise-Women Puzzle

Semantics of KT45^n (continued)

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- $x \Vdash E_G \phi$ iff for each $i \in G$, $x \Vdash K_i \phi$.
- $x \Vdash C_G \phi$ iff for each $k \geq 1$, we have $x \Vdash E_G{}^k \phi$.
- $x \Vdash D_G \phi$ iff for each $y \in W$, we have $y \Vdash \phi$, whenever $R_i(x, y)$ for all $i \in G$.

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation: The Wise Women Puzzle
Modal Logic KT45^n
Models of KT45^n
Formulation of Wise-Women Puzzle

Formulation of Wise-Women Puzzle

Setup

- Wise woman i has red hat: p_i
- Wise woman i knows that wise woman j has a red hat: $K_i \, p_j$

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation: The Wise Women Puzzle
Modal Logic KT45^n
Models of KT45^n
Formulation of Wise-Women Puzzle

Formulation of Wise-Women Puzzle

Initial situation

$$\begin{aligned}
\Gamma \;=\; \{ \quad & C(p_1 \vee p_2 \vee p_3), \\
& C(p_1 \to K_2 p_1), C(\neg p_1 \to K_2 \neg p_1), \\
& C(p_1 \to K_3 p_1), C(\neg p_1 \to K_3 \neg p_1), \\
& C(p_2 \to K_1 p_2), C(\neg p_2 \to K_1 \neg p_2), \\
& C(p_2 \to K_3 p_2), C(\neg p_2 \to K_3 \neg p_2), \\
& C(p_3 \to K_1 p_3), C(\neg p_2 \to K_1 \neg p_3), \\
& C(p_3 \to K_2 p_3), C(\neg p_2 \to K_2 \neg p_3) \}
\end{aligned}$$

Review of Modal Logic
Some Modal Logics
Natural Deduction in Modal Logic
Knowledge in Multi-Agent Systems

Motivation: The Wise Women Puzzle
Modal Logic KT45^n
Models of KT45^n
Formulation of Wise-Women Puzzle

Announcements

First wise woman says "No"

$$C(\neg K_1 p_1 \wedge \neg K_1 \neg p_1)$$

Second wise woman says "No"

$$C(\neg K_2 p_2 \wedge \neg K_2 \neg p_2)$$

Review of Modal Logic	Motivation: The Wise Women Puzzle
Some Modal Logics	Modal Logic KT45^n
Natural Deduction in Modal Logic	Models of KT45^n
Knowledge in Multi-Agent Systems	**Formulation of Wise-Women Puzzle**

First Attempt

$$\Gamma, C(\neg K_1 p_1 \wedge \neg K_1 \neg p_1), C(\neg K_2 p_2 \wedge \neg K_2 \neg p_2) \vdash K_3 p_3$$

Review of Modal Logic	**Motivation: The Wise Women Puzzle**
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First Attempt

$$\Gamma, C(\neg K_1 p_1 \wedge \neg K_1 \neg p_1), C(\neg K_2 p_2 \wedge \neg K_2 \neg p_2) \vdash K_3 p_3$$

Problem

This does not take time into account. The second announcement can take the first announcement into account.

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Solution

Prove separately:

Entailment 1 :

$$\Gamma, C(\neg K_1 p_1 \wedge \neg K_1 \neg p_1) \vdash C(p_2 \vee p_3)$$

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Solution

Prove separately:

Entailment 1 :

$$\Gamma, C(\neg K_1 p_1 \wedge \neg K_1 \neg p_1) \vdash C(p_2 \vee p_3)$$

Entailment 2 :

$$\Gamma, C(p_2 \vee p_3), C(\neg K_2 p_2 \wedge \neg K_2 \neg p_2) \vdash K_3 p_3$$

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Solution

Prove separately:

Entailment 1 :

$$\Gamma, C(\neg K_1 p_1 \wedge \neg K_1 \neg p_1) \vdash C(p_2 \vee p_3)$$

Entailment 2 :

$$\Gamma, C(p_2 \vee p_3), C(\neg K_2 p_2 \wedge \neg K_2 \neg p_2) \vdash K_3 p_3$$

Proof

Through natural deduction in KT45^n.

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Next Week

- Hoare Logic