Initial Segment Complexity for Measures First Results and Open Problems

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Kolmogorov Complexity

Prefix-Free: $\mathbf{K}$; Plain: $\mathbf{C}$.

Properties of $\mathbf{C}, \mathbf{K}$ by choice of universal machines: For all binary strings $\mathbf{x}$, $\mathbf{C}(\mathbf{x}) \leq \mathbf{K}(\mathbf{x})$ and $\mathbf{C}(\mathbf{0}^{|\mathbf{x}|}) \leq \mathbf{C}(\mathbf{x})$ and $\mathbf{K}(\mathbf{0}^{|\mathbf{x}|}) \leq \mathbf{K}(\mathbf{x})$; here $\mathbf{0}^{\mathbf{n}}$ stands for $\mathbf{n}$.

Used for characterising algorithmically random sets: Martin-Löf random sets, Kolmogorov random sets (= Martin-Löf random relative halting problem).

Used for characterising triviality ($\mathbf{C}$-trivial = Recursive; $\mathbf{K}$-trivial is subclass of superlow sets).

Goal: Do similar characterisations for measures.

History

Bhojraj as well as Nies and Scholz 2019 investigated initial segment complexity of measures within the framework of quantum information theory; the current work carries some results over which have easier proofs in the classical world. They also studied Solovay tests and strong Solovay tests.

Schnorr and later Kautz 1991 and Porter 2015 studied discrete measures, that is, measures which are possibly infinite convex combinations of Dirac measures (atoms). These measures include in particular all $\mathbf{C}$-trivial and $\mathbf{K}$-trivial measures as studied below.

Measures on Cantor Space

Measure of Binary Strings: $\mu(\mathbf{x}) = \mu(\{\mathbf{A} : \mathbf{x} \preceq \mathbf{A}\})$.

Initial Segment Complexity

$$\mathbf{C}(\mu \upharpoonright \mathbf{n}) = \sum_{\mathbf{x} \in \{0,1\}^{\mathbf{n}}} \mu(\mathbf{x}) \cdot \mathbf{C}(\mathbf{x})$$

Triviality: A measure μ is $\mathbf{C}$-trivial iff there is a constant $\mathbf{c}$ with $\mathbf{C}(\mu \upharpoonright \mathbf{n}) \leq \mathbf{C}(\mathbf{n}) + \mathbf{c}$ for all $\mathbf{n}$. Similarly one defines $\mathbf{K}$-trivial measures.

For this recall that a set $\mathbf{A}$ is trivial (with respect to $\mathbf{C}, \mathbf{K}$) iff the Kolmogorov complexity of a prefix $\mathbf{x}$ equals, up to constant, that of $\mathbf{0}^{|\mathbf{x}|}$.

Martin-Löf Absolutely Continuous: A measure μ is Martin-Löf absolutely continuous iff for every Martin-Löf test $\mathbf{G}_0, \mathbf{G}_1, \mathbf{G}_2, \ldots$, $\mu(\mathbf{G}_\mathbf{n})$ must go to $\mathbf{0}$ for $\mathbf{n} \to \infty$.

Examples

1. The uniform measure is Martin-Löf absolutely continuous.

2. Let $\delta_{\mathbf{A}}$ is a Dirac measure $(\delta_{\mathbf{A}}(\{\mathbf{A}\}) = \mathbf{1})$. Then the following holds:

- $\delta_{\mathbf{A}}$ is Martin-Löf absolutely continuous iff $\mathbf{A}$ is Martin-Löf random;
- $\delta_{\mathbf{A}}$ is $\mathbf{K}$-trivial iff $\mathbf{A}$ is $\mathbf{K}$-trivial;
- $\delta_{\mathbf{A}}$ is $\mathbf{C}$-trivial iff $\mathbf{A}$ is $\mathbf{C}$-trivial (that is equivalent to recursive).

3. Mauldin and Montecino defined a probability space of measures and Culver showed that this probability space is recursive. Randomly drawn measures according to this probability space are never discrete and do not have atoms; such measures are Martin-Löf absolutely continuous.

Convex Combinations

If $\mu_{\mathbf{k}}$ are measures and $\sum_{\mathbf{k}} \alpha_{\mathbf{k}} = \mathbf{1}$ then $\nu = \sum_{\mathbf{k}} \alpha_{\mathbf{k}} \cdot \mu_{\mathbf{k}}$ is called a convex combination of these measures. Schnorr called a measure which is a convex combination of atoms “discrete”.

Fact

Convex combinations of Martin-Löf absolutely continuous measures are Martin-Löf absolutely continuous.
Finite convex combinations of $\mathbf{C}$-trivial measures are $\mathbf{C}$-trivial and finite convex combinations of $\mathbf{K}$-trivial measures are $\mathbf{K}$-trivial.

Corollary

The Turing degree of $\mathbf{C}$-trivial and $\mathbf{K}$-trivial measures can be arbitrary.

Hint: $\delta_{\emptyset} \cdot \mathbf{a} + \delta_{\mathbb{N}} \cdot (\mathbf{1} - \mathbf{a})$ is trivial and has Turing degree of $\mathbf{a}$.

Strong Triviality I

The following results are stated for $\mathbf{C}$ but also hold for $\mathbf{K}$.

Theorem. If μ is $\mathbf{C}$-trivial then μ is of the possibly infinite convex combination $\sum_{\mathbf{k}} \alpha_{\mathbf{k}} \cdot \delta_{\mathbf{A_k}}$ of Dirac measures of $\mathbf{C}$-trivial sets $\mathbf{A_k}$.

Proof Idea. For each constant $\mathbf{c}$ there is a constant $\mathbf{d}$ such that for all lengthes $\mathbf{n}$ there are at most $\mathbf{d}$ strings $\mathbf{x} \in \{\mathbf{0}, \mathbf{1}\}^{\mathbf{n}}$ with $\mathbf{C}(\mathbf{x}) \leq \mathbf{C}(\mathbf{n}) + \mathbf{c}$. Let $\mathbf{r_{n,c}}$ be their combined measure for length $\mathbf{n}$. If there is a constant $\mathbf{q} > \mathbf{0}$ such that, for all $\mathbf{c}$ there are infinitely many $\mathbf{n}$ with $\mathbf{r_{n,c}} < \mathbf{1} - \mathbf{q}$ then μ is not $\mathbf{C}$-trivial.

For each $\mathbf{c}$ there are up to $\mathbf{d}$ sets $A_1, \ldots, A_d$ such that, for infinitely many $\mathbf{n}$, all strings of measure up to $\mathbf{C}(\mathbf{n}) + \mathbf{c}$ of length $\mathbf{n}$ are prefixes of $\mathbf{A_1}, \ldots, \mathbf{A_d}$. These form a part $\mathbf{b} \sum_{\mathbf{e} \leq \mathbf{d}} \delta_{\mathbf{A_e}} \cdot \mathbf{a_e}$ of combined measure $\mathbf{r_c}$ and limit $\mathbf{r_c} = \mathbf{1}$. Thus μ is the convex combination of Dirac measures.

Strong Triviality II

Definition
Let $\mathbf{b_k}$ be the $\mathbf{C}$-triviality constant of $\mathbf{C}$-trivial $\mathbf{A_k}$. If $\sum_{\mathbf{k}} \alpha_{\mathbf{k}} \cdot \mathbf{b_k} < \infty$ and $\sum_{\mathbf{k}} \alpha_{\mathbf{k}} = 1$ then one calls the convex combination $\mu = \sum_{\mathbf{k}} \alpha_{\mathbf{k}} \cdot \delta_{\mathbf{A_k}}$ a strongly $\mathbf{C}$-trivial measure.

Theorem
1. Strongly $\mathbf{C}$-trivial measures are $\mathbf{C}$-trivial;
2. Some measures are $\mathbf{C}$-trivial but not strongly $\mathbf{C}$-trivial.
3. There are infinite convex combinations of Dirac measures of $\mathbf{C}$-trivial sets which are not $\mathbf{C}$-trivial as a measure.

Proof of 1. $\mathbf{C}(\mu \upharpoonright \mathbf{n}) \leq \sum_{\mathbf{k},\mathbf{x}:\mathbf{x} \preceq \mathbf{A_k}, |\mathbf{x}|=\mathbf{n}} \mathbf{a_k} \cdot \mathbf{C}(\mathbf{x}) \leq$
$\sum_{\mathbf{k}} \mathbf{a_k} \cdot (\mathbf{C}(\mathbf{n}) + \mathbf{b_k}) \leq \mathbf{C}(\mathbf{n}) + \sum_{\mathbf{k}} \mathbf{a_k} \cdot \mathbf{b_k} \leq \mathbf{C}(\mathbf{n}) + \mathbf{Const}$.
Thus μ is $\mathbf{C}$ trivial. Similarly for $\mathbf{K}$.

Examples

Example for Strongly $\mathbf{C}$-trivial measure. Let $\mathbf{A_k} = \{\mathbf{k}\}$ and let $\mathbf{b_k}$ be the triviality constant of $\mathbf{A_k}$. Now let $\mathbf{a} = \sum_\mathbf{k} \mathbf{k}^{-2}/(\mathbf{b_k}+\mathbf{1})$ and $\mathbf{a_k} = \mathbf{k}^{-2}/(\mathbf{a}\cdot(\mathbf{b_k}+\mathbf{1}))$. Now $\sum_\mathbf{k} \mathbf{a_k} = \mathbf{1}$ and $\sum_\mathbf{k} \mathbf{a_k}\cdot\mathbf{b_k} \leq \sum_\mathbf{k} \mathbf{k}^{-2} < \infty$. Thus the measure $\mu = \sum_\mathbf{k} \mathbf{a_k}\cdot\delta_{\mathbf{A_k}}$ is strongly $\mathbf{C}$-trivial.

Example for measure satisfying 3. Let $\mathbf{A_k}$ be a finite set with minimum $\mathbf{k}$ and maximum $\mathbf{2k}$ such that $\mathbf{C}(\mathbf{A_k}(\mathbf{0})\ldots\mathbf{A_k}(\mathbf{2k})) \geq \mathbf{k}/\mathbf{3}$. Furthermore, let $\mathbf{a} = \sum_\mathbf{k} \mathbf{2}^{-\mathbf{K}(\mathbf{k})}$ and $\mathbf{a_k} = \mathbf{2}^{-\mathbf{k}}/\mathbf{a}$. As the prefix-free Kolmogorov complexity of a string can be very small compared to $\mathbf{k}$, there are infinitely many $\mathbf{k}$ where $\mathbf{C}(\mathbf{A_k}(\mathbf{0})\ldots\mathbf{A_k}(\mathbf{2k}))\cdot\mathbf{a_k} \geq \sqrt{\mathbf{k}}$ and thus the measure $\mu = \sum_\mathbf{k} \mathbf{a_k}\delta_{\mathbf{A_k}}$ is not $\mathbf{C}$-trivial. Note that trivial measures satisfy $\mathbf{C}(\mu \upharpoonright \mathbf{2k}) \leq \log \mathbf{k} + \mathbf{Const}$ for all $\mathbf{k}$.

Does C-trivial imply K-trivial? - I

Lemma. There is a constant $\mathbf{d}$ such that, for all $\mathbf{A}, \mathbf{b}$, if $\mathbf{A}$ is $\mathbf{C}$-trivial with constant $\mathbf{b}$ then $\mathbf{A}$ is $\mathbf{K}$-trivial with constant $\mathbf{2b + d}$.

Proof. By work of Chaitin and Nies there are at most $\mathbf{b^2 \cdot 2^b}$ strings $\mathbf{x}$ of length $\mathbf{n}$ with $\mathbf{C(x) \leq C(n) + b}$. There is an r.e. tree $\mathbf{T_b}$ which enumerates all strings $\mathbf{x}$ such that there is a string $\mathbf{y}$ of length $\mathbf{2^{|x|+1}}$ with prefix $\mathbf{x \preceq y}$ and $\mathbf{C(z) \leq \log|z| + b + 1}$ for all $\mathbf{z \preceq y}$.

The tree $\mathbf{T_b}$ has on all levels at most $\mathbf{b^2 \cdot 2^b}$ strings. These can be described in a prefix-free way on length $\mathbf{n}$ with $\mathbf{K(n) + 2b + d}$ bits, one uses $\mathbf{K(n)}$ bits to describe $\mathbf{n}$ and uses then $\mathbf{K(b)}$ bits to describe $\mathbf{b}$ followed by $\mathbf{2\log(b) + b}$ bits to describe the sequence number in the enumeration of $\mathbf{T_b}$ of the string in $\mathbf{T_b \cap \{0,1\}^n}$ desired. Thus $\mathbf{A}$ is $\mathbf{K}$-trivial with constant $\mathbf{d + 2b}$ or less.

Does C-trivial imply K-trivial? - II

Theorem

Every strongly $\mathbf{C}$-trivial measure is strongly $\mathbf{K}$-trivial.

Proof

μ is strongly $\mathbf{C}$-trivial $\Rightarrow$
$\mu = \sum_k a_k \cdot A_k$ such that $\sum_k a_k \cdot b_k$ is convergent where b_k is $\mathbf{C}$-triviality constant of A_k $\Rightarrow$
$\mu = \sum_k a_k \cdot A_k$ such that $\sum_k a_k \cdot (2b_k + d)$ is convergent where $2b_k + d$ is bound on the $\mathbf{K}$-triviality constant of A_k $\Rightarrow$
μ is strongly $\mathbf{K}$-trivial.

Here note that $\sum_k a_k = 1$ and that finite sums of convergent sums are convergent. As all terms in the sums are nonnegative, order of summation can be modified.

Does C-trivial imply K-trivial? - III

Open Question

Is every $\mathbf{C}$-trivial measure $\mathbf{K}$-trivial?

The following three statements are equivalent and sufficient for giving a positive answer.

- There is a constant $\mathbf{c}$ such that for all $\mathbf{n}$ and $\mathbf{x} \in \{0,1\}^{\mathbf{n}}$ it holds that $\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{n}) \leq \mathbf{2} \cdot (\mathbf{C}(\mathbf{x}) - \mathbf{C}(\mathbf{n})) + \mathbf{c}$;
- For every $\mathbf{c}'$ there is a $\mathbf{d}'$ so that all $\mathbf{n}$ and $\mathbf{x} \in \{0,1\}^{\mathbf{n}}$ satisfy $\mathbf{C}(\mathbf{x}) - \mathbf{C}(\mathbf{n}) \leq \mathbf{c}' \Rightarrow \mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{n}) \leq \mathbf{d}'$;
- There is a constant $\mathbf{c}''$ bounding $\mathbf{K}(\mathbf{C}(\mathbf{n})|\mathbf{n}, \mathbf{K}(\mathbf{n}))$ for all $\mathbf{n}$.

Further Results on Triviality

Theorem
For every unbounded increasing function $\mathbf{f}$ which is recursively approximable from above, there is a measure μ satisfying $\mathbf{C}(\mu \upharpoonright \mathbf{n}) \leq^+ \mathbf{C}(\mathbf{n}) + \mathbf{f}(\mathbf{n})$ and $\mathbf{K}(\mu \upharpoonright \mathbf{n}) \leq^+ \mathbf{K}(\mathbf{n}) + \mathbf{f}(\mathbf{n})$ for all $\mathbf{n}$, without any atoms.

Idea. Construct r.e. set with very thin complement and then measure which is uniform on all supersets of that r.e. set.

Theorem
If Γ is a truth-table reduction and $\mu(\mathbf{x}) = \nu(\{\mathbf{A} : \Gamma^{\mathbf{A}} \succeq \mathbf{x}\})$ then ν $\mathbf{C}$-trivial implies μ $\mathbf{C}$-trivial; however, if Γ is only a Turing reduction which might be undefined on $\emptyset$ which is not an atom of ν, then it can happen that ν is $\mathbf{C}$-trivial but μ not. The same holds for $\mathbf{K}$ in place of $\mathbf{C}$.

Comparing Measures

Definition: A measure ρ passes a μ-ML-test $\mathbf{G_0}, \mathbf{G_1}, \ldots$ iff $\lim_{\mathbf{n}} \rho(\mathbf{G_n}) = \mathbf{0}$. ρ Martin-Löf absolute continuous in μ iff ρ passes every μ-ML-test.

Theorem: The following are equivalent for a measure ρ:

- ρ is a possibly infinite convex combination of Dirac measures given by $\mathbf{K}$-trivial sets;
- ρ is Martin-Löf absolutely continuous in a $\mathbf{K}$-trivial measure μ;
- ρ is Martin-Löf absolutely continuous in every measure ν which is an infinite convex combination of all Dirac measures of $\mathbf{K}$-trivial atoms, not leaving out any possible $\mathbf{K}$-trivial atom.

The analogous result holds when $\mathbf{K}$ is replaced by $\mathbf{C}$.

Solovay Tests

A test $\{G_m\}$ for a measure μ is a Solovay test iff $\sum_m \mu(G_m) < \infty$; Solovay tests generalise Martin-Löf tests. Furthermore, if the G_m are effectively generated by a finite set of strings (output by some recursive function), then one calls it a strong Solovay test. Bhojraj investigated these tests in the quantum setting and investigated the initial segment complexity of measures ν passing all strong Solovay tests μ.

If a measure ρ does not pass all strong Solovay tests (with respect to uniform measure λ) then its initial segment complexity $K(\rho \upharpoonright n)$ is staying below $n - \delta f(n)$ for some $\delta > 0$ and recursive function f with $\sum_n 2^{-f(n)} < \infty$.

Open Question

If a measure passes all strong Solovay tests, does it also pass all Solovay tests?

Absolutely Continuous

Call μ Kolmogorov random iff $\exists \mathbf{c}\, \exists^\infty \mathbf{n}\, [\mathbf{C}(\mu \upharpoonright \mathbf{n}) \geq \mathbf{n} - \mathbf{c}]$.

Examples: The uniform measure and the Dirac measure of a Kolmogorov random set are both Kolmogorov random.

Theorem

There are measures μ, ν such that

- μ is Martin-Löf absolutely continuous but there are infinitely many $\mathbf{n}$ with $\mathbf{K}(\mu \upharpoonright \mathbf{n}) \leq \mathbf{n}$ and
- ν is not Martin-Löf absolutely continuous but satisfies $\lim_{\mathbf{n} \to \infty} \mathbf{K}(\nu \upharpoonright \mathbf{n}) - \mathbf{n} = \infty$.

Theorem

If μ is Kolmogorov random then μ is Martin-Löf absolutely continuous.

Further Open Questions

1. Are finite convex combinations of Kolmogorov random measures also Kolmogorov random?

2. Given two Kolmogorov random sets $\mathbf{A}, \mathbf{B}$, is there a constant $\mathbf{c}$ such that there are infinitely many $\mathbf{n}$ satisfying both $\mathbf{C}(\mathbf{A} \upharpoonright \mathbf{n}) \geq \mathbf{n} - \mathbf{c}$ and $\mathbf{C}(\mathbf{B} \upharpoonright \mathbf{n}) \geq \mathbf{n} - \mathbf{c}$?

A “yes” for the first question implies a “yes” for the second question; if $\mathbf{A} \oplus \mathbf{B}$ is Kolmogorov random, the second question is answered affirmatively.

The same topic could be investigated for the notion that there is a $\mathbf{c}$ such that there are infinitely many $\mathbf{n}$ with $\mathbf{K}(\mathbf{A} \upharpoonright \mathbf{n}) \geq \mathbf{n} + \mathbf{K}(\mathbf{n}) - \mathbf{c}$ (strongly Chaitin random).

Conclusion

Initial segment complexity of measures is defined analogously to the initial segment complexity of sets: $\mathbf{C}(\mu \upharpoonright \mathbf{n}) = \sum_{\mathbf{x} \in \{\mathbf{0},\mathbf{1}\}^{\mathbf{n}}} \mu(\mathbf{x}) \cdot \mathbf{C}(\mathbf{x})$. One of the goal of this definition was to keep it compatible with the notion for sets in the case of Dirac measures.

The lower end (trivial measures) and the upper end (Kolmogorov random measures; Martin-Löf absolutely continuous measures) have been investigated.

Some natural conjectures (closure of Kolmogorov random measures under convex combination; implication $\mathbf{C}$-trivial $\Rightarrow$ $\mathbf{K}$-trivial for measures) could not be resolved, though these conjectures are linked to not that likely but naturally formulated conjectures about basic Kolmogorov complexity properties.