Theory of Computation 5
Combining Languages

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Repetition 1

If $(\mathbf{Q}, \mathbf{\Sigma}, \delta, \mathbf{s}, \mathbf{F})$ is a non-deterministic finite automaton (nfa) then δ has a set of values (not always single value), that is, for $\mathbf{p} \in \mathbf{Q}$ and $\mathbf{a} \in \mathbf{\Sigma}$ there can be several $\mathbf{q} \in \mathbf{Q}$ such that the nfa can go from $\mathbf{p}$ to $\mathbf{q}$ on symbol $\mathbf{a}$.

A run of an nfa on a word $\mathbf{a_1 a_2 \ldots a_n}$ is a sequence $\mathbf{q_0 q_1 q_2 \ldots q_n} \in \mathbf{Q}^*$ such that $\mathbf{q_0} = \mathbf{s}$ and $\mathbf{q_{m+1}} \in \delta(\mathbf{q_m}, \mathbf{a_{m+1}})$ for all $\mathbf{m} < \mathbf{n}$.

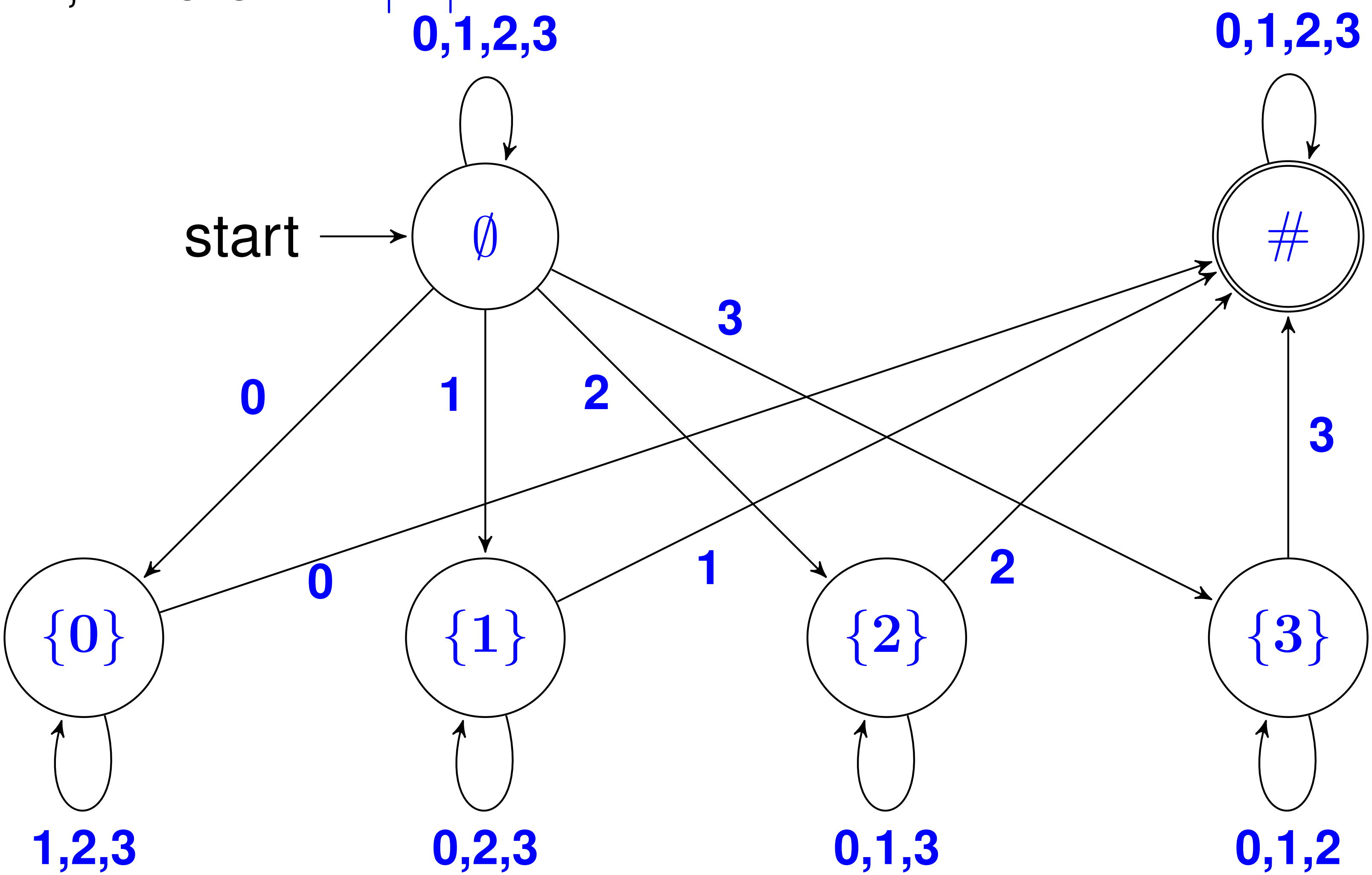
If $\mathbf{q_n} \in \mathbf{F}$ then the run is “accepting” else the run is “rejecting”.

The nfa accepts a word $\mathbf{w}$ iff it has an accepting run on $\mathbf{w}$; this is also the case if there exist other rejecting runs.

δ as relation: $(\mathbf{p}, \mathbf{a}, \mathbf{q}) \in \delta$ iff nfa can go on $\mathbf{a}$ from $\mathbf{p}$ to $\mathbf{q}$.
δ as set-valued function: $\delta(\mathbf{p}, \mathbf{a}) = \{\mathbf{q} :$ nfa can go on $\mathbf{a}$ from $\mathbf{p}$ to $\mathbf{q}\}$.

Repetition 2

The language $\{w : \text{some letter appears twice}\}$ has an nfa with $n + 2$ states while a dfa needs $2^n + 1$ states; here for $n = 4$, where $n = |\Sigma|$.



Repetition 3

Given an nfa, one let for given state $\mathbf{q}$ and symbol $\mathbf{a}$ the set $\delta(\mathbf{q},\mathbf{a})$ denote all states $\mathbf{q}'$ to which the nfa can transit from $\mathbf{q}$ on symbol $\mathbf{a}$.

Theorem 4.5 [Büchi; Rabin and Scott]
For each nfa $(\mathbf{Q},\mathbf{\Sigma},\delta,\mathbf{s},\mathbf{F})$ with $\mathbf{n} = |\mathbf{Q}|$ states, there is an equivalent dfa $(\{\mathbf{Q}' : \mathbf{Q}' \subseteq \mathbf{Q}\},\mathbf{\Sigma},\delta',\{\mathbf{s}\},\mathbf{F}')$ with $\mathbf{2^n}$ states such that $\mathbf{F}' = \{\mathbf{Q}' \subseteq \mathbf{Q} : \mathbf{Q}' \cap \mathbf{F} \neq \emptyset\}$ and

$$\forall \mathbf{Q}' \subseteq \mathbf{Q}\, \forall \mathbf{a} \in \mathbf{\Sigma}\, [\delta'(\mathbf{Q}',\mathbf{a}) \;=\; \bigcup_{\mathbf{q}' \in \mathbf{Q}} \delta(\mathbf{q}',\mathbf{a}) \;=\; \{\mathbf{q}'' \in \mathbf{Q} : \exists \mathbf{q}' \in \mathbf{Q}'\, [\mathbf{q}'' \in \delta(\mathbf{q}',\mathbf{a})]\}].$$

As the number of states is often overshooting, it is good to minimise the resulting automaton with the algorithm of Myhill and Nerode.

Repetition 4

The following statements are all equivalent to "L is regular":

(a) L is generated by a regular expression;

(b) L is generated by a regular grammar;

(c) L is recognised by a determinisitic finite automaton;

(d) L is recognised by a non-determinisitic finite automaton;

(e) L and $\Sigma^* - L$ both satisfy the Block Pumping Lemma;

(f) L satsifies Jaffe's Matching Pumping Lemma;

(g) L has only finitely many derivatives.

Product Automata

Let $(\mathbf{Q_1}, \mathbf{\Sigma}, \delta_\mathbf{1}, \mathbf{s_1}, \mathbf{F_1})$ and $(\mathbf{Q_2}, \mathbf{\Sigma}, \delta_\mathbf{2}, \mathbf{s_2}, \mathbf{F_2})$ be dfas which recognise $\mathbf{L_1}$ and $\mathbf{L_2}$, respectively.

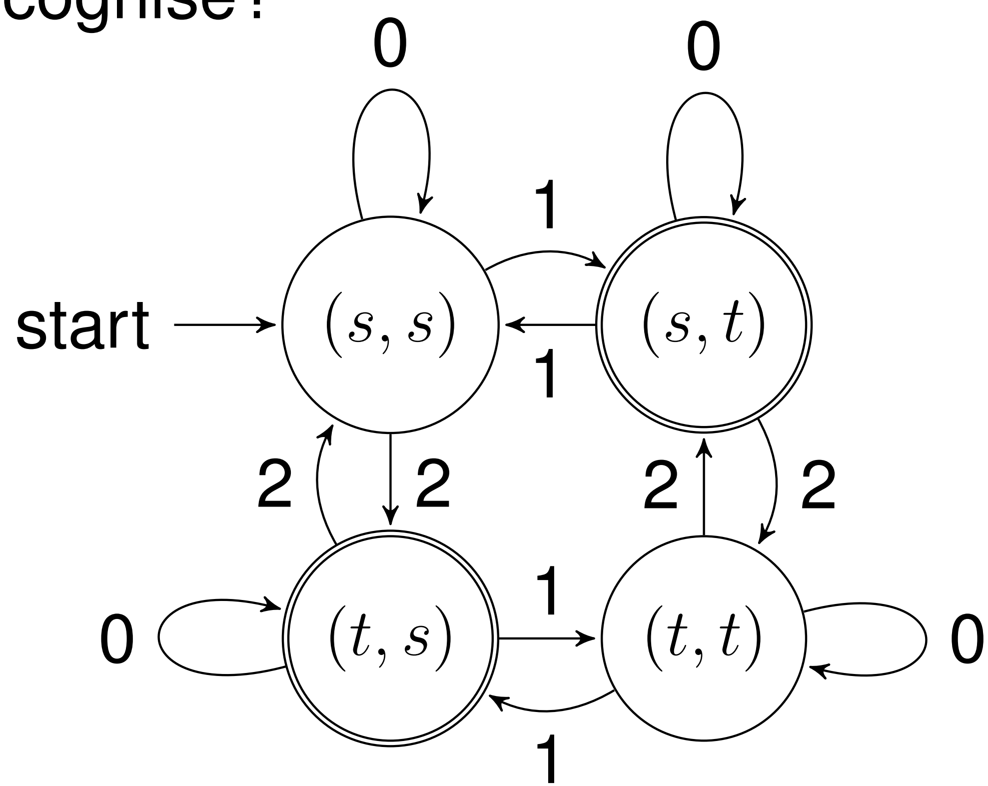
Consider $(\mathbf{Q_1} \times \mathbf{Q_2}, \mathbf{\Sigma}, \delta_\mathbf{1} \times \delta_\mathbf{2}, (\mathbf{s_1}, \mathbf{s_2}), \mathbf{F})$ with $(\delta_\mathbf{1} \times \delta_\mathbf{2})((\mathbf{q_1}, \mathbf{q_2}), \mathbf{a}) = (\delta_\mathbf{1}(\mathbf{q_1}, \mathbf{a}), \delta_\mathbf{2}(\mathbf{q_2}, \mathbf{a}))$. This automaton is called a product automaton and one can choose $\mathbf{F}$ such that it recognises the union or intersection or difference of the respective languages.

Union: $\mathbf{F} = (\mathbf{F_1} \times \mathbf{Q_2}) \cup (\mathbf{Q_1} \times \mathbf{F_2})$;
Intersection: $\mathbf{F} = \mathbf{F_1} \times \mathbf{F_2} = (\mathbf{F_1} \times \mathbf{Q_2}) \cap (\mathbf{Q_1} \times \mathbf{F_2})$;
Difference: $\mathbf{F} = \mathbf{F_1} \times (\mathbf{Q_2} - \mathbf{F_2})$;
Symmetric Difference:
$\mathbf{F} = (\mathbf{F_1} \times (\mathbf{Q_2} - \mathbf{F_2})) \cup ((\mathbf{Q_1} - \mathbf{F_1}) \times \mathbf{F_2})$.

Example

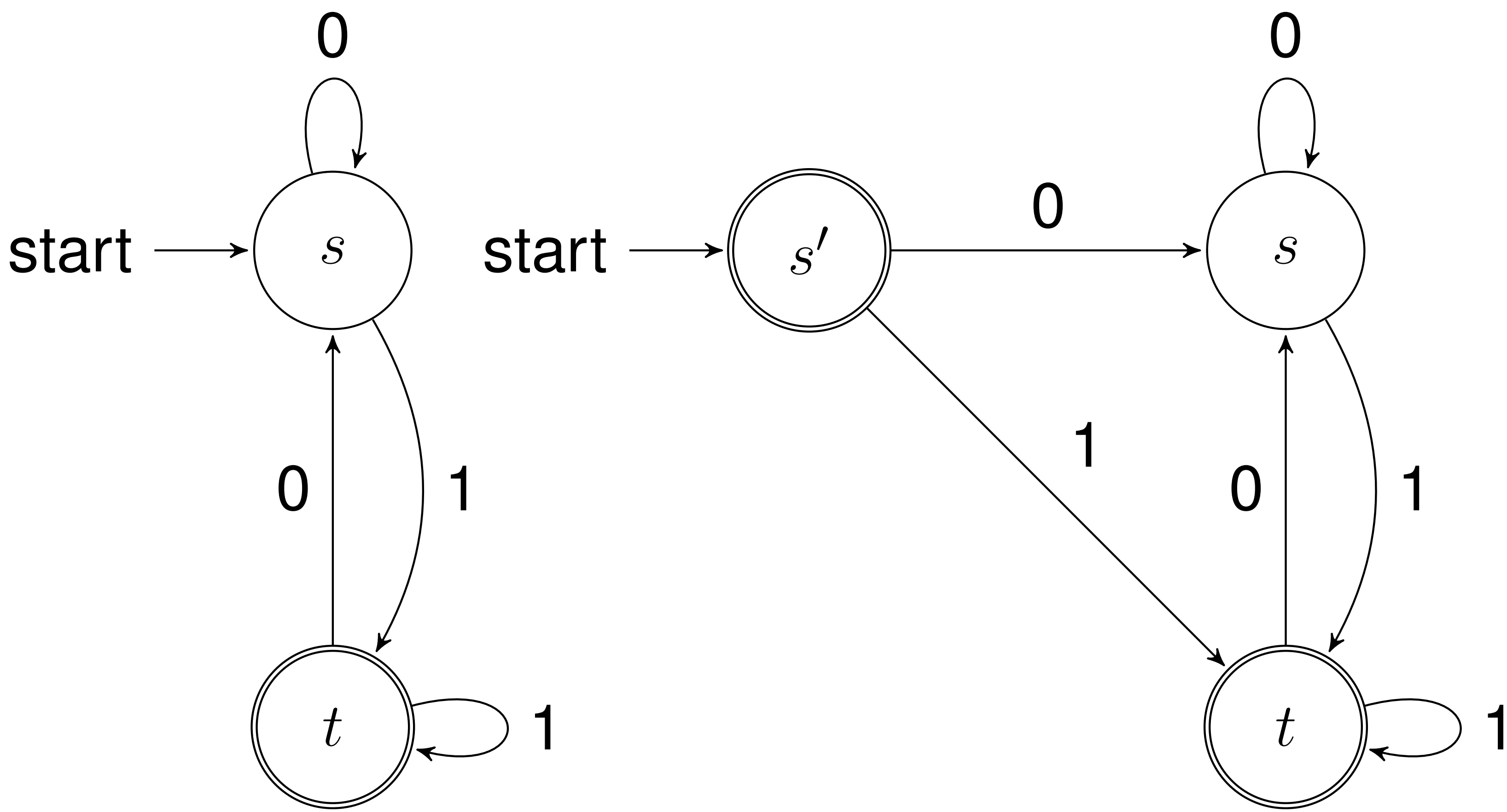
For $\mathbf{a} = \mathbf{1}, \mathbf{2}$, let automaton $(\{\mathbf{s}, \mathbf{t}\}, \{\mathbf{0}, \mathbf{1}, \mathbf{2}\}, \delta_{\mathbf{a}}, \mathbf{s}, \{\mathbf{s}\})$ recognise when there is an even number of $\mathbf{a}$; if input $\mathbf{b}$ equals $\mathbf{a}$ then state is changed else state remains unchanged.

Quiz: Which Boolean combination does this product automaton recognise?



Kleene Star

Assume $(\mathbf{Q}, \mathbf{\Sigma}, \delta, \mathbf{s}, \mathbf{F})$ is an nfa recognising $\mathbf{L}$. Now $\mathbf{L}^*$ is recognised by $(\mathbf{Q} \cup \{\mathbf{s}'\}, \mathbf{\Sigma}, \delta', \mathbf{s}', \{\mathbf{s}'\} \cup \mathbf{F})$ where $\delta'(\mathbf{s}', \mathbf{a}) = \delta(\mathbf{s}, \mathbf{a})$ and $\delta'(\mathbf{p}, \mathbf{a}) = \delta(\mathbf{p}, \mathbf{a})$ for $\mathbf{p} \in \mathbf{Q} - \mathbf{F}$ and $\delta'(\mathbf{p}, \mathbf{a}) = \delta(\mathbf{p}, \mathbf{a}) \cup \delta(\mathbf{s}, \mathbf{a})$ for $\mathbf{p} \in \mathbf{F}$.



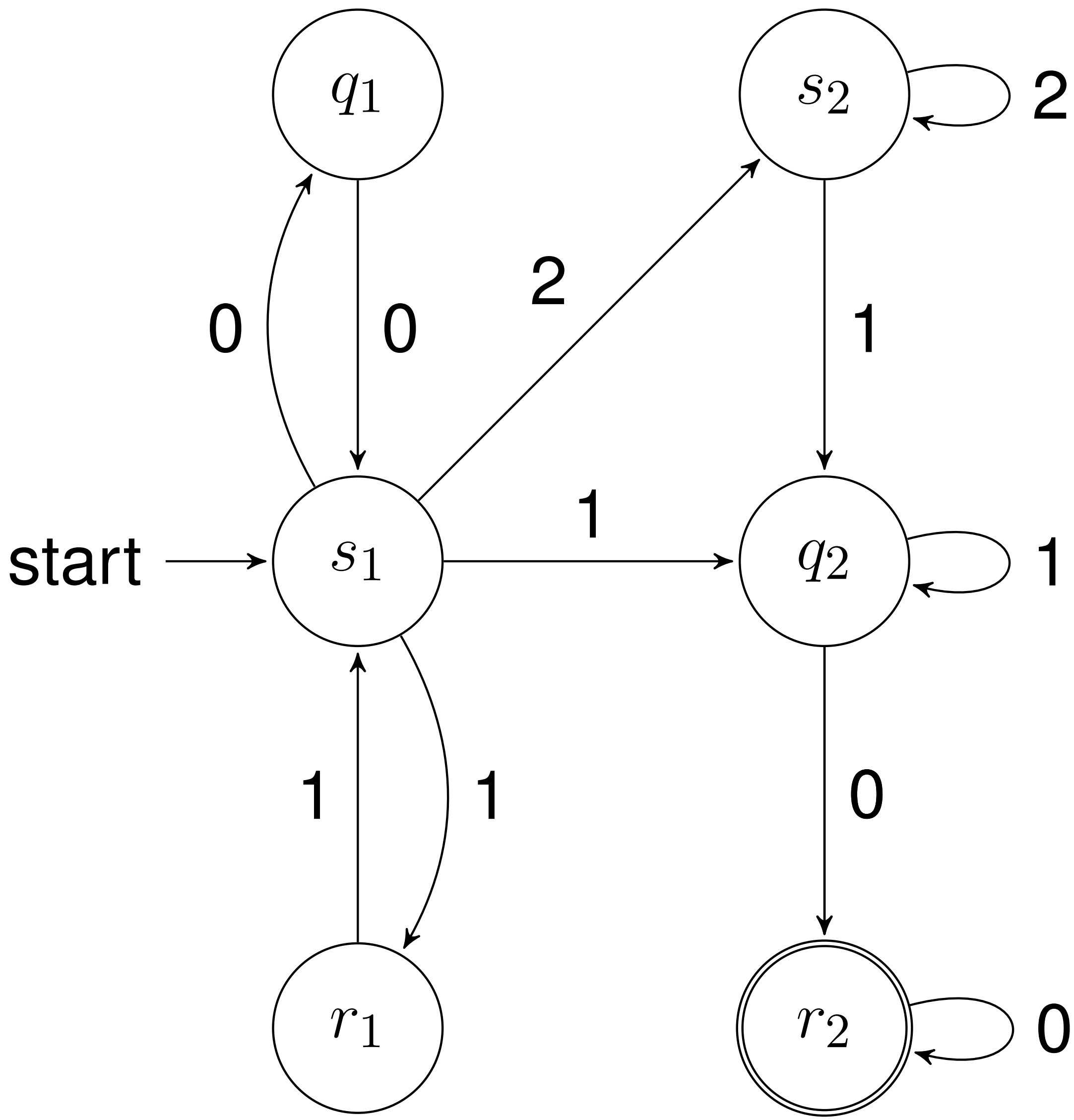
Concatenation

Assume $(\mathbf{Q_1}, \mathbf{\Sigma}, \delta_\mathbf{1}, \mathbf{s_1}, \mathbf{F_1})$ and $(\mathbf{Q_2}, \mathbf{\Sigma}, \delta_\mathbf{2}, \mathbf{s_2}, \mathbf{F_2})$ are nfas recognising $\mathbf{L_1}$ and $\mathbf{L_2}$ with $\mathbf{Q_1} \cap \mathbf{Q_2} = \emptyset$ and assume $\varepsilon \notin \mathbf{L_2}$. Now $(\mathbf{Q_1} \cup \mathbf{Q_2}, \mathbf{\Sigma}, \delta, \mathbf{s_1}, \mathbf{F_2})$ recognises $\mathbf{L_1} \cdot \mathbf{L_2}$ where $(\mathbf{p}, \mathbf{a}, \mathbf{q}) \in \delta$ whenever $(\mathbf{p}, \mathbf{a}, \mathbf{q}) \in \delta_\mathbf{1} \cup \delta_\mathbf{2}$ or ($\mathbf{p} \in \mathbf{F_1}$ and $(\mathbf{s_2}, \mathbf{a}, \mathbf{q}) \in \delta_\mathbf{2}$).

If $\mathbf{L_2}$ contains ε then one can consider the union of $\mathbf{L_1}$ and $\mathbf{L_1} \cdot (\mathbf{L_2} - \{\varepsilon\})$.

Example

$\mathbf{L_1 \cdot L_2}$ with $\mathbf{L_1 = \{00, 11\}^*}$ and $\mathbf{L_2 = 2^* 1^+ 0^+}$.



Exercise 5.3

The previous slides give upper bounds on the size of the dfa for a union, intersection, difference and symmetric difference as $\mathbf{n^2}$ states, provided that the original two dfas have at most $\mathbf{n}$ states.

Give the corresponding bounds for nfas: If $\mathbf{L}$ and $\mathbf{H}$ are recognised by nfas having at most $\mathbf{n}$ states each, how many states does one need at most for an nfa recognising (a) the union $\mathbf{L} \cup \mathbf{H}$, (b) the intersection $\mathbf{L} \cap \mathbf{H}$, (c) the difference $\mathbf{L} - \mathbf{H}$ and (d) the symmetric difference $(\mathbf{L} - \mathbf{H}) \cup (\mathbf{H} - \mathbf{L})$?

Give the bounds in terms of “linear”, “quadratic” and “exponential”. Explain your bounds.

Sample Automata

Exercise 5.4
Let $\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Construct a (not necessarily complete) dfa recognising the language $\Sigma \cdot \{aa : a \in \Sigma\}^* \cap \{aaaaa : a \in \Sigma\}^*$. It is not needed to give a full table for the dfa, but a general schema and an explanation how it works.

Exercise 5.5
Make an nfa for the intersection of the following languages: $\{0, 1, 2\}^* \cdot \{001\} \cdot \{0, 1, 2\}^* \cdot \{001\} \cdot \{0, 1, 2\}^*$; $\{001, 0001, 2\}^*$; $\{0, 1, 2\}^* \cdot \{00120001\} \cdot \{0, 1, 2\}^*$.

Exercise 5.6
Make an nfa for the union $L_0 \cup L_1 \cup L_2$ with $L_a = \{0, 1, 2\}^* \cdot \{aa\} \cdot \{0, 1, 2\}^* \cdot \{aa\} \cdot \{0, 1, 2\}^*$ for $a \in \{0, 1, 2\}$.

Exercise 5.7

Consider two context-free grammars with terminals Σ, disjoint non-terminals N_1 and N_2, start symbols $S_1 \in N_1$ and $S_2 \in N_2$ and rule sets P_1 and P_2 which generate L and H, respectively. Explain how to form from these a new context-free grammar for

(a) $L \cup H$,
(b) $L \cdot H$ and
(c) L^*.

Write down the context-free grammars for $\{0^n 1^{2n} : n \in \mathbb{N}\}$ and $\{0^n 1^{3n} : n \in \mathbb{N}\}$ and form the grammars for union, concatenation and star explicitly.

Example 5.8

The language $\{0\}^* \cdot \{1^n 2^n : n \in \mathbb{N}\}$ is context-free.

Grammar $(\{S, T\}, \{0, 1, 2\}, P, S)$ with P be given by $S \to 0S|T|\varepsilon$ and $T \to 1T2|\varepsilon$.

The language $\{0^n 1^n : n \in \mathbb{N}\} \cdot \{2\}^*$ is context-free.

$L = \{0^n 1^n 2^n : n \in \mathbb{N}\}$ is not context-free but the intersection of the two above.

The complement of L is the union of $\{0^n 1^m 2^k : n < k\}$, $\{0^n 1^m 2^k : n > k\}$, $\{0^n 1^m 2^k : m < k\}$, $\{0^n 1^m 2^k : m > k\}$, $\{0^n 1^m 2^k : n < m\}$, $\{0^n 1^m 2^k : n > m\}$ and $\{0, 1, 2\}^* \cdot \{10, 20, 21\} \cdot \{0, 1, 2\}^*$.
Each of these languages is context-free. Grammar for the first of them: $S \to 0S2|S2|T2, T \to 1T|\varepsilon$. The union is also context-free. Hence L has a context-free complement.

So context-free languages are neither closed under intersection nor under complement.

Context-Free Intersects Regular

Theorem 5.9
If $\mathbf{L}$ is context-free and $\mathbf{H}$ is regular then $\mathbf{L} \cap \mathbf{H}$ is context-free.

Construction.
Let $(\mathbf{N}, \mathbf{\Sigma}, \mathbf{P}, \mathbf{S})$ be a context-free grammar generating $\mathbf{L}$ with every rule being either $\mathbf{A} \rightarrow \mathbf{w}$ or $\mathbf{A} \rightarrow \mathbf{BC}$ with $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbf{N}$ and $\mathbf{w} \in \mathbf{\Sigma}^*$.

Let $(\mathbf{Q}, \mathbf{\Sigma}, \delta, \mathbf{s}, \mathbf{F})$ be a dfa recognising $\mathbf{H}$.

Let $\mathbf{S}' \notin \mathbf{Q} \times \mathbf{N} \times \mathbf{Q}$ and make the following new grammar $(\mathbf{Q} \times \mathbf{N} \times \mathbf{Q} \cup \{\mathbf{S}'\}, \mathbf{\Sigma}, \mathbf{R}, \mathbf{S}')$ with rules $\mathbf{R}$:
$\mathbf{S}' \rightarrow (\mathbf{s}, \mathbf{S}, \mathbf{q})$ for all $\mathbf{q} \in \mathbf{F}$;
$(\mathbf{p}, \mathbf{A}, \mathbf{q}) \rightarrow (\mathbf{p}, \mathbf{B}, \mathbf{r})(\mathbf{r}, \mathbf{C}, \mathbf{q})$ for all rules $\mathbf{A} \rightarrow \mathbf{BC}$ in $\mathbf{P}$ and all $\mathbf{p}, \mathbf{q}, \mathbf{r} \in \mathbf{Q}$;
$(\mathbf{p}, \mathbf{A}, \mathbf{q}) \rightarrow \mathbf{w}$ for all rules $\mathbf{A} \rightarrow \mathbf{w}$ in $\mathbf{P}$ with $\delta(\mathbf{p}, \mathbf{w}) = \mathbf{q}$.

Exercises 5.10 and 5.11

Recall that the language $\mathbf{L}$ of all words which contain as many $\mathbf{0}$s as $\mathbf{1}$s is context-free; a grammar for it is $(\{\mathbf{S}\}, \{\mathbf{0}, \mathbf{1}\}, \{\mathbf{S} \to \mathbf{SS}|\varepsilon|\mathbf{0S1}|\mathbf{1S0}\}, \mathbf{S})$.

Exercise 5.10

Construct a context-free grammar for $\mathbf{L} \cap (\mathbf{001}^+)^*$.

Exercise 5.11

Construct a context-free grammar for $\mathbf{L} \cap \mathbf{0}^*\mathbf{1}^*\mathbf{0}^*\mathbf{1}^*$.

Context-Sensitive and Concatenation

Let $\mathbf{L_1}$ and $\mathbf{L_2}$ be context-sensitive languages not containing ε. Let $(\mathbf{N_1}, \mathbf{\Sigma}, \mathbf{P_1}, \mathbf{S_1})$ and $(\mathbf{N_2}, \mathbf{\Sigma}, \mathbf{P_2}, \mathbf{S_2})$ be two context-senstive grammers generating $\mathbf{L_1}$ and $\mathbf{L_2}$, respectively, where $\mathbf{N_1} \cap \mathbf{N_2} = \emptyset$ and where each rule $\mathbf{l} \to \mathbf{r}$ satisfies $|\mathbf{l}| \leq |\mathbf{r}|$ and $\mathbf{l} \in \mathbf{N_e^+}$ for the respective $\mathbf{e} \in \{\mathbf{1}, \mathbf{2}\}$. Let $\mathbf{S} \notin \mathbf{N_1} \cup \mathbf{N_2} \cup \mathbf{\Sigma}$.

Now $(\mathbf{N_1} \cup \mathbf{N_2} \cup \{\mathbf{S}\}, \mathbf{\Sigma}, \mathbf{P_1} \cup \mathbf{P_2} \cup \{\mathbf{S} \to \mathbf{S_1 S_2}\}, \mathbf{S})$ generates $\mathbf{L_1} \cdot \mathbf{L_2}$.

If $\mathbf{v} \in \mathbf{L_1}$ and $\mathbf{w} \in \mathbf{L_2}$ then $\mathbf{S} \Rightarrow \mathbf{S_1 S_2} \Rightarrow^* \mathbf{v S_2} \Rightarrow^* \mathbf{vw}$. Furthermore, the first rule has to be $\mathbf{S} \Rightarrow \mathbf{S_1 S_2}$ and from then onwards, each rule has on the left side either $\mathbf{l} \in \mathbf{N_1^+}$ so that it applies to the part generated from $\mathbf{S_1}$ or it has in the left side $\mathbf{l} \in \mathbf{N_2^+}$ so that $\mathbf{l}$ is in the part of the word generated from $\mathbf{S_2}$. Hence every intermediate word $\mathbf{z}$ in the derivation is of the form $\mathbf{xy} = \mathbf{z}$ with $\mathbf{S_1} \Rightarrow^* \mathbf{x}$ and $\mathbf{S_2} \Rightarrow^* \mathbf{y}$.

Context-Sensitive and Kleene-star

Let $(\mathbf{N_1}, \mathbf{\Sigma}, \mathbf{P_1}, \mathbf{S_1})$ and $(\mathbf{N_2}, \mathbf{\Sigma}, \mathbf{P_2}, \mathbf{S_2})$ be context-sensitive grammars for $\mathbf{L} - \{\varepsilon\}$ with $\mathbf{N_1} \cap \mathbf{N_2} = \emptyset$ and all rules $\mathbf{l} \rightarrow \mathbf{r}$ satisfying $|\mathbf{l}| \leq |\mathbf{r}|$ and $\mathbf{l} \in \mathbf{N_1^+}$ or $\mathbf{l} \in \mathbf{N_2^+}$, respectively. Let $\mathbf{S}, \mathbf{S'}$ be symbols not in $\mathbf{N_1} \cup \mathbf{N_2} \cup \mathbf{\Sigma}$.

Now consider $(\mathbf{N_1} \cup \mathbf{N_2} \cup \{\mathbf{S}, \mathbf{S'}\}, \mathbf{\Sigma}, \mathbf{P}, \mathbf{S})$ where $\mathbf{P}$ contains the rules $\mathbf{S} \rightarrow \mathbf{S'} | \varepsilon$ and $\mathbf{S'} \rightarrow \mathbf{S_1 S_2 S'} \mid \mathbf{S_1 S_2} \mid \mathbf{S_1}$ plus all rules in $\mathbf{P_1} \cup \mathbf{P_2}$.

This grammar generates $\mathbf{L^*}$.

Context-Sensitive and Intersection

Theorem.
The intersection of two context-sensitive languages is context-sensitive.

Construction.
Let $(\mathbf{N_k}, \mathbf{\Sigma}, \mathbf{P_k}, \mathbf{S})$ be grammars for $\mathbf{L_1}$ and $\mathbf{L_2}$. Now make a new non-terminal set $\mathbf{N} = (\mathbf{N_1} \cup \mathbf{\Sigma} \cup \{\#\}) \times (\mathbf{N_2} \cup \mathbf{\Sigma} \cup \{\#\})$ with start symbol $\binom{\mathbf{S}}{\mathbf{S}}$ and following types of rules:
(a) Rules to generate and manage space;
(b) Rules to generate a word $\mathbf{v}$ in the upper row;
(c) Rules to generate a word $\mathbf{w}$ in the lower row;
(d) Rules to convert a string from $\mathbf{N}$ into $\mathbf{v}$ provided that the upper components and lower components of the string are both $\mathbf{v}$.

Type of Rules

(a): $\binom{\mathbf{S}}{\mathbf{S}} \rightarrow \binom{\mathbf{S}}{\mathbf{S}}\binom{\#}{\#}$ for producing space; $\binom{\mathbf{A}}{\mathbf{B}}\binom{\#}{\mathbf{C}} \rightarrow \binom{\#}{\mathbf{B}}\binom{\mathbf{A}}{\mathbf{C}}$ and $\binom{\mathbf{A}}{\mathbf{C}}\binom{\mathbf{B}}{\#} \rightarrow \binom{\mathbf{A}}{\#}\binom{\mathbf{B}}{\mathbf{C}}$ for space management.

(b) and (c): For each rule in $\mathbf{P_1}$, for example, for $\mathbf{AB} \rightarrow \mathbf{CDE} \in \mathbf{P_1}$, and all symbols $\mathbf{F}, \mathbf{G}, \mathbf{H}, \ldots$ in $\mathbf{N_2}$, one has the corresponding rule $\binom{\mathbf{A}}{\mathbf{F}}\binom{\mathbf{B}}{\mathbf{G}}\binom{\#}{\mathbf{H}} \rightarrow \binom{\mathbf{C}}{\mathbf{F}}\binom{\mathbf{D}}{\mathbf{G}}\binom{\mathbf{E}}{\mathbf{H}}$. So rules in $\mathbf{P_1}$ are simulated in the upper half and rules in $\mathbf{P_2}$ are simulated in the lower half and they use up $\#$ if the left side is shorter than the right one.

(d): Each rule $\binom{\mathbf{a}}{\mathbf{a}} \rightarrow \mathbf{a}$ for $\mathbf{a} \in \mathbf{\Sigma}$ is there to convert a matching pair $\binom{\mathbf{a}}{\mathbf{a}}$ from $\mathbf{\Sigma} \times \mathbf{\Sigma}$ (a nonterminal) to $\mathbf{a}$ (a terminal).

Grammar for $0^n1^n2^n$ with $n > 0$

Grammar L_1: $S \to S2|0S1|01$.
Grammar L_2: $S \to 0S|1S2|12$.

Grammar for Intersection.
$N = \{\binom{A}{B} : A, B \in \{S, 0, 1, 2, \#\}\}$.
Rules where A, B, C stand for any members of
$\{S, 0, 1, 2, \#\}$: $\binom{S}{S} \to \binom{S}{S}\binom{\#}{\#}$;
$\binom{A}{B}\binom{\#}{C} \to \binom{\#}{B}\binom{A}{C}$; $\binom{A}{C}\binom{B}{\#} \to \binom{A}{\#}\binom{B}{C}$;
$\binom{S}{A}\binom{\#}{B} \to \binom{S}{A}\binom{2}{B}$; $\binom{S}{A}\binom{\#}{B}\binom{\#}{C} \to \binom{0}{A}\binom{S}{B}\binom{1}{C}$;
$\binom{S}{A}\binom{\#}{B} \to \binom{0}{A}\binom{1}{B}$;
$\binom{A}{S}\binom{B}{\#} \to \binom{A}{0}\binom{B}{S}$; $\binom{A}{S}\binom{B}{\#}\binom{C}{\#} \to \binom{A}{1}\binom{B}{S}\binom{C}{2}$;
$\binom{A}{S}\binom{B}{\#} \to \binom{A}{1}\binom{B}{2}$;
$\binom{0}{0} \to 0$; $\binom{1}{1} \to 1$; $\binom{2}{2} \to 2$.

Deriving 001122

$$\binom{S}{S} \Rightarrow^* \binom{S}{S}\binom{\#}{\#}\binom{\#}{\#}\binom{\#}{\#}\binom{\#}{\#}\binom{\#}{\#} \Rightarrow \binom{S}{S}\binom{2}{\#}\binom{\#}{\#}\binom{\#}{\#}\binom{\#}{\#}\binom{\#}{\#} \Rightarrow^*$$

$$\binom{S}{S}\binom{\#}{\#}\binom{\#}{\#}\binom{\#}{\#}\binom{\#}{\#}\binom{2}{\#} \Rightarrow \binom{S}{S}\binom{2}{\#}\binom{\#}{\#}\binom{\#}{\#}\binom{\#}{\#}\binom{2}{\#} \Rightarrow^*$$

$$\binom{S}{S}\binom{\#}{\#}\binom{\#}{\#}\binom{\#}{\#}\binom{2}{\#}\binom{2}{\#} \Rightarrow \binom{0}{S}\binom{S}{\#}\binom{1}{\#}\binom{\#}{\#}\binom{2}{\#}\binom{2}{\#} \Rightarrow$$

$$\binom{0}{S}\binom{S}{\#}\binom{\#}{\#}\binom{1}{\#}\binom{2}{\#}\binom{2}{\#} \Rightarrow \binom{0}{S}\binom{0}{\#}\binom{1}{\#}\binom{1}{\#}\binom{2}{\#}\binom{2}{\#} \Rightarrow$$

$$\binom{0}{0}\binom{0}{S}\binom{1}{\#}\binom{1}{\#}\binom{2}{\#}\binom{2}{\#} \Rightarrow \binom{0}{0}\binom{0}{0}\binom{1}{S}\binom{1}{\#}\binom{2}{\#}\binom{2}{\#} \Rightarrow$$

$$\binom{0}{0}\binom{0}{0}\binom{1}{1}\binom{1}{S}\binom{2}{2}\binom{2}{\#} \Rightarrow \binom{0}{0}\binom{0}{0}\binom{1}{1}\binom{1}{S}\binom{2}{\#}\binom{2}{2} \Rightarrow$$

$$\binom{0}{0}\binom{0}{0}\binom{1}{1}\binom{1}{1}\binom{2}{2}\binom{2}{2} \Rightarrow^* 001122.$$

Exercises 5.14 and 5.17

Exercise 5.14

Let $\mathbf{L} = \{\mathbf{0^n 1^n 2^n} : \mathbf{n} \in \mathbb{N}\}$ and construct a context-sensitive grammar for $\mathbf{L}^*$.

Exercise 5.17

Consider the language $\mathbf{L} = \{\mathbf{00}\} \cdot \{\mathbf{0,1,2,3}\}^* \cup \{\mathbf{1,2,3}\} \cdot \{\mathbf{0,1,2,3}\}^* \cup \{\mathbf{0,1,2,3}\}^* \cdot \{\mathbf{02,03,13,10,20,30,21,31,32}\} \cdot \{\mathbf{0,1,2,3}\}^* \cup \{\varepsilon\} \cup \{\mathbf{01^n 2^n 3^n} : \mathbf{n} \in \mathbb{N}\}$.

Which versions of the Pumping Lemma does it satisfy:

- Regular Pumping Lemma (with / without bounds);
- Context-Free Pumping Lemma (with / without bounds);
- Block Pumping Lemma (for regular languages)?

Determine the exact position of $\mathbf{L}$ in the Chomsky hierarchy.

Mirror Images

Define $(\mathbf{a_1 a_2 \ldots a_n})^{\mathbf{mi}} = \mathbf{a_n \ldots a_2 a_1}$ as the mirror image of a string.

It follows from the definition of context-free and context-sensitive, that if $\mathbf{L}$ is context-free / context-sensitive so is $\mathbf{L}^{\mathbf{mi}}$. This can be achieved by replacing every rule $\mathbf{l} \to \mathbf{r}$ by $\mathbf{l}^{\mathbf{mi}} \to \mathbf{r}^{\mathbf{mi}}$.

For example, the mirror image of the language of the words $\mathbf{0^n 1^{3n+3}}$ is given by language of the words $\mathbf{1^{3n+3} 0^n}$. While $\mathbf{L}$ is generated by a context-free grammar with one non-terminal $\mathbf{S}$ and rules $\mathbf{S} \to \mathbf{0S111} \,|\, \mathbf{111}$, $\mathbf{L}^{\mathbf{mi}}$ is then generated by a similar grammar with the rules $\mathbf{S} \to \mathbf{111S0} \,|\, \mathbf{111}$.

Exercise 5.18

Recall that $\mathbf{x^{mi}}$ is the mirror image of $\mathbf{x}$, so $\mathbf{(01001)^{mi} = 10010}$. Furthermore, $\mathbf{L^{mi} = \{x^{mi} : x \in L\}}$. Show the following two statements:

(a) If an nfa with $\mathbf{n}$ states recognises $\mathbf{L}$ then there is also an nfa with up to $\mathbf{n+1}$ states recognising $\mathbf{L^{mi}}$.

(b) Find the smallest nfas which recognise $\mathbf{L = 0^*(1^* \cup 2^*)}$ as well as $\mathbf{L^{mi}}$.

Palindromes

The members of the language $\{x \in \Sigma^* : x = x^{mi}\}$ are called palindromes. A palindrome is a word or phrase which looks the same from both directions.

An example is the German name “OTTO”; furthermore, when ignoring spaces and punctuation marks, a famous palindrome is the phrase “A man, a plan, a canal: Panama.” This palindrome was found by Leigh Mercer (1893-1977), a British hobby-writer, who created lots of palindromes.

The grammar with the rules $S \to aSa|aa|a|\varepsilon$ with a ranging over all members of Σ generates all palindromes; so for $\Sigma = \{0, 1, 2\}$ the rules of the grammar would be $S \to 0S0 \,|\, 1S1 \,|\, 2S2 \,|\, 00 \,|\, 11 \,|\, 22 \,|\, 0 \,|\, 1 \,|\, 2 \,|\, \varepsilon$. Therefore the set of palindromes is context-free.

The set of palindromes is not regular.

Exercises

Exercise 5.20

Let $\mathbf{w} \in \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}, \mathbf{9}\}^*$ be a palindrome of even length and $\mathbf{n}$ be its decimal value. Prove that $\mathbf{n}$ is a multiple of $\mathbf{11}$. Note that it is essential that the length is even, as for odd length there are counter examples (like $\mathbf{111}$ and $\mathbf{202}$).

Exercise 5.21

Given a context-free grammar for a language $\mathbf{L}$, is there also one for $\mathbf{L} \cap \mathbf{L}^{\mathbf{mi}}$? If so, explain how to construct the grammar; if not, provide a counter example where $\mathbf{L}$ is context-free but $\mathbf{L} \cap \mathbf{L}^{\mathbf{mi}}$ is not.

Exercises

Exercise 5.22

Is the following statement true or false? Prove your answer:
Given a language $\mathbf{L}$, the language $\mathbf{L} \cap \mathbf{L}^{\mathbf{mi}}$ equals to $\{\mathbf{w} \in \mathbf{L} : \mathbf{w}$ is a palindrome$\}$.

Exercise 5.23

Let $\mathbf{L} = \{\mathbf{w} \in \{\mathbf{0}, \mathbf{1}, \mathbf{2}\}^* : \mathbf{w} = \mathbf{w}^{\mathbf{mi}}\}$ and consider
$\mathbf{H} = \mathbf{L} \cap \{\mathbf{012}, \mathbf{210}, \mathbf{00}, \mathbf{11}, \mathbf{22}\}^* \cap (\{\mathbf{0}, \mathbf{1}\}^* \cdot \{\mathbf{1}, \mathbf{2}\}^* \cdot \{\mathbf{0}, \mathbf{1}\}^*)$.
This is the intersection of a context-free and regular language and thus context-free. Construct a context-free grammar for $\mathbf{H}$.

Exercises

In the following, one considers regular expressions consisting of the symbol $\mathbf{L}$ of palindromes over $\{0, 1, 2\}$ and the mentioned operations. What is the most difficult level in the hierarchy "regular, linear, context-free, context-sensitive" such expressions can generate. It can be used that $\{10^i10^j10^k1 : i \neq j, i \neq k, j \neq k\}$ is not context-free.

Exercise 5.24: Expressions containing $\mathbf{L}$ and $\cup$ and finite sets.

Exercise 5.25: Expressions containing $\mathbf{L}$ and $\cup$ and $\cdot$ and Kleene star and finite sets.

Exercise 5.26: Expressions containing $\mathbf{L}$ and $\cup$ and $\cdot$ and and $\cap$ and Kleene star and finite sets.

Exercise 5.27: Expressions containing $\mathbf{L}$ and $\cdot$ and set difference and Kleene star and finite sets.

Homomorphism

Example

Let $\mathbf{ascii(Year\ 2021) = 5965617220323032 31}$ represent each letter of "$\mathbf{Year\ 2021}$" by its two-digit hexadecimal ASCII representation.

Definition 5.28

A homomorphism is a mapping $\mathbf{h}$ with domain $\mathbf{\Sigma^*}$ for some alphabet $\mathbf{\Sigma}$ which preserves concatenation: $\mathbf{h(v \cdot w) = h(v) \cdot h(w)}$.

Proposition 5.29

The homomorphism is determined by the images of the single letters and $\mathbf{h(w) = h(a_1) \cdot h(a_2) \cdot \ldots \cdot h(a_n)}$ for a word $\mathbf{w = a_1 a_2 \ldots a_n}$; $\mathbf{h(\varepsilon) = \varepsilon}$.

Quiz

What is $\mathbf{ascii(Year\ 1819)}$ for above homomorphism $\mathbf{ascii}$?

Exercises 5.30 and 5.31

Count the number of homomorphisms and list them; explain why there are not more. Two homomorphisms are the same iff they have the same values $\mathbf{h(0), h(1), h(2), h(3)}$. Here they take values from $\mathbf{4}^*$.

Exercise 5.30

How many homomorphisms $\mathbf{h}$ satisfy $\mathbf{h(012) = 44444}$, $\mathbf{h(102) = 444444}$, $\mathbf{h(00) = 44444}$ and $\mathbf{h(3) = 4}$?

Exercise 5.31

How many homomorphisms $\mathbf{h}$ satisfy $\mathbf{h(012) = 44444}$, $\mathbf{h(102) = 44444}$, $\mathbf{h(0011) = 444444}$ and $\mathbf{h(3) = 44}$?

Homomorphic Images

Theorem 5.32
The homomorphic images of regular and context-free languages are regular and context-free, respectively.

Construction
Given a homomorphism $\mathbf{h}$, replace in any rule of a given regular / context-free grammar every terminal $\mathbf{a}$ by the word $\mathbf{h(a)}$; these replacements only occur on the right side of the rules. The type of the grammar remains unchanged.

For a proof that $\mathbf{S} \Rightarrow^* \mathbf{w}$ in the original grammar iff $\mathbf{S} \Rightarrow \mathbf{h(w)}$ in the new grammar, one shows by induction for a derivation $\mathbf{S} \Rightarrow \mathbf{v_1} \Rightarrow \ldots \Rightarrow \mathbf{v_n} \Rightarrow \mathbf{w}$ translates into $\mathbf{h(S)} \Rightarrow \mathbf{h(v_1)} \Rightarrow \ldots \Rightarrow \mathbf{h(v_n)} \Rightarrow \mathbf{h(w)}$ where $\mathbf{h}$ is extended by letting $\mathbf{h(A)} = \mathbf{A}$ for all non-terminals $\mathbf{A}$. The converse also holds.

Example 5.33

One can apply the homomorphisms also directly to regular expressions using the rules $\mathbf{h(L \cup H) = h(L) \cup h(H)}$, $\mathbf{h(L \cdot H) = h(L) \cdot h(H)}$ and $\mathbf{h(L^*) = (h(L))^*}$. Thus one can move a homomorphism into the inner parts (which are the finite sets used in the regular expression) and then apply the homomorphism there.

So for the language $\mathbf{(\{0,1\}^* \cup \{0,2\}^*) \cdot \{33\}^*}$ and the homomorphism which maps each symbol $\mathbf{a}$ to $\mathbf{aa}$, one obtains the language $\mathbf{(\{00,11\}^* \cup \{00,22\}^*) \cdot \{3333\}^*}$.

Context-Senstive Languages

Theorem 5.38
Every recursively enumerable language (= language generated by some grammar) is the homomorphic image of a context-sensitive language.

The idea is that if some grammar generates $(N, \{1, 2, \ldots, k\}, P, S)$ for L, one can make a new grammar for a context-sensitive language H such that for all $w \in \{1, 2, \ldots, k\}^*$, $w \in L$ iff $w \cdot 0^\ell \in H$ for some ℓ. These additional 0 will be used to make words longer so that in the new grammar, all rules $l \to r$ satisfy $|l| \leq |r|$ which is obtained sufficiently many 0 on the right side and by making rules for 0 to swap with other symbols to move right.

Images of Homomorphisms

Determine $\mathbf{h(L)}$ for the following languages:

(a) $\{\mathbf{0},\mathbf{1},\mathbf{2}\}^*$;

(b) $\{\mathbf{00},\mathbf{11},\mathbf{22}\}^* \cap \{\mathbf{000},\mathbf{111},\mathbf{222}\}^*$;

(c) $(\{\mathbf{00},\mathbf{11}\}^* \cup \{\mathbf{00},\mathbf{22}\}^* \cup \{\mathbf{11},\mathbf{22}\}^*) \cdot \{\mathbf{011222}\}$;

(d) $\{\mathbf{w} \in \{0,1\}^* : \mathbf{w}$ has more $\mathbf{1}$s than it has $\mathbf{0}$s$\}$.

Exercise 5.40
$\mathbf{h}$ is given as $\mathbf{h(0) = 1}$, $\mathbf{h(1) = 22}$, $\mathbf{h(2) = 333}$.

Exercise 5.41
$\mathbf{h}$ is given as $\mathbf{h(0) = 3}$, $\mathbf{h(1) = 4}$, $\mathbf{h(2) = 334433}$.

Exercise 5.42

Let a homomorphism $\mathbf{h}:\{\mathbf{0},\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{5},\mathbf{6},\mathbf{7},\mathbf{8},\mathbf{9}\}^* \rightarrow \{\mathbf{0},\mathbf{1},\mathbf{2},\mathbf{3}\}^*$ be given by the equations $\mathbf{h}(\mathbf{0}) = \mathbf{0}$, $\mathbf{h}(\mathbf{1}) = \mathbf{h}(\mathbf{4}) = \mathbf{h}(\mathbf{7}) = \mathbf{1}$, $\mathbf{h}(\mathbf{2}) = \mathbf{h}(\mathbf{5}) = \mathbf{h}(\mathbf{8}) = \mathbf{2}$, $\mathbf{h}(\mathbf{3}) = \mathbf{h}(\mathbf{6}) = \mathbf{h}(\mathbf{9}) = \mathbf{3}$. Interpret the images of $\mathbf{h}$ as quaternary numbers $($numbers of base four, so $\mathbf{12321}$ represents $\mathbf{1}$ times two hundred fifty six plus $\mathbf{2}$ times sixty four plus $\mathbf{3}$ times sixteen plus $\mathbf{2}$ times four plus $\mathbf{1})$. Prove the following:

- Every quaternary number is the image of a decimal number without leading zeroes;
- A decimal number $\mathbf{w}$ has leading zeroes iff the quaternary number $\mathbf{h}(\mathbf{w})$ has leading zeroes;
- A decimal number $\mathbf{w}$ is a multiple of three iff the quaternary number $\mathbf{h}(\mathbf{w})$ is a multiple of three.

Exercise 5.43

Consider only homomorphisms
$\mathbf{h} : \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}, \mathbf{9}\}^* \rightarrow \{\mathbf{0}, \mathbf{1}\}^*$ such that

- $\mathbf{h}(\mathbf{w})$ has leading zeroes iff $\mathbf{w}$ has;
- $\mathbf{h}(\mathbf{0}) = \mathbf{0}$;
- the range of $\mathbf{h}$ is $\{\mathbf{0}, \mathbf{1}\}^*$.

For each of $\mathbf{p} = \mathbf{2}, \mathbf{3}, \mathbf{5}$, answer the following question: Can one choose $\mathbf{h}$ such that, in addition, $\mathbf{w}$ is a multiple of $\mathbf{p}$ iff $\mathbf{h}(\mathbf{w})$ is, as a binary number, a multiple of $\mathbf{p}$?

If $\mathbf{h}$ can be chosen as desired then list this $\mathbf{h}$ else prove that such a homomorphism $\mathbf{h}$ cannot exist.

Inverse Homomorphism

Description 5.46
Let $\mathbf{h}$ have domain $\mathbf{\Sigma}^*$ and the set $\mathbf{h^{-1}(L) = \{w \in \Sigma^* : h(w) \in L\}}$ is called the inverse image of $\mathbf{h}$. $\mathbf{h^{-1}}$ satisfies the following rules:

(a) $\mathbf{h^{-1}(L) \cap h^{-1}(H) = h^{-1}(L \cap H)}$;

(b) $\mathbf{h^{-1}(L) \cup h^{-1}(H) = h^{-1}(L \cup H)}$;

(c) $\mathbf{h^{-1}(L) \cdot h^{-1}(H) \subseteq h^{-1}(L \cdot H)}$;

(d) $\mathbf{h^{-1}(L)^* \subseteq h^{-1}(L^*)}$.

Here $\mathbf{L = H = \{0\}}$ and $\mathbf{h(a) = aa}$ for all $\mathbf{a \in \Sigma}$ implies $\mathbf{h^{-1}(L) = h^{-1}(H) = \emptyset}$, $\mathbf{(h^{-1}(L))^* = \{\varepsilon\}}$, $\mathbf{h^{-1}(L \cdot H) = \{0\}}$ and $\mathbf{h^{-1}(L^*) = \{0\}^*}$.

Theorem 5.47 and Exercise 5.48

Theorem 5.47
If $\mathbf{L}$ is on level $\mathbf{k}$ of the Chomsky hierarchy and $\mathbf{h}$ is an homomorphism then $\mathbf{h^{-1}(L)}$ is on level $\mathbf{k}$ of the Chomsky hierarchy.

Construction for the regular case: If $(\mathbf{Q}, \mathbf{\Gamma}, \gamma, \mathbf{s}, \mathbf{F})$ is a dfa recognising $\mathbf{L}$ and $\mathbf{h} : \mathbf{\Sigma}^* \rightarrow \mathbf{\Gamma}^*$ is an homomorphism then $(\mathbf{Q}, \mathbf{\Sigma}, \delta, \mathbf{s}, \mathbf{F})$ is a dfa recognising $\mathbf{h^{-1}(L)}$ where, for every $\mathbf{q} \in \mathbf{Q}$ and $\mathbf{a} \in \mathbf{\Sigma}$, $\delta(\mathbf{q}, \mathbf{a}) = \gamma(\mathbf{q}, \mathbf{h}(\mathbf{a}))$.

Exercise 5.48
Let $\mathbf{h} : \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}^* \rightarrow \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}\}^*$ be given by $\mathbf{h(0) = 00}$, $\mathbf{h(1) = 012}$, $\mathbf{h(2) = 123}$ and $\mathbf{h(3) = 1}$ and let $\mathbf{L}$ consist of all words containing exactly five $\mathbf{0}$s and at least one $\mathbf{2}$. Construct a complete dfa recognising $\mathbf{h^{-1}(L)}$.