Subdividing Alpha Complex

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Abstract. Given two simplicial complexes $\mathcal{C}_1$ and $\mathcal{C}_2$ embedded in Euclidean space $\mathbb{R}^d$, $\mathcal{C}_1$ *subdivides* $\mathcal{C}_2$ if (i) $\mathcal{C}_1$ and $\mathcal{C}_2$ have the same underlying space, and (ii) every simplex in $\mathcal{C}_1$ is contained in a simplex in $\mathcal{C}_2$. In this paper we present a method to compute a set of weighted points whose alpha complex subdivides a given simplicial complex.
Following this, we also show a simple method to approximate a given polygonal object with a set of balls via computing the subdividing alpha complex of the boundary of the object. The approximation is robust and is able to achieve a union of balls whose Hausdorff distance to the object is less than a given positive real number ϵ.

1 Introduction

The notion of alpha complexes is defined by Edelsbrunner [6, 10] and since then it has been widely applied in various fields such as computer graphics, solid modeling, computational biology, computational geometry and topology [7, 8]. In this paper, we propose a simple algorithm to compute the alpha complex that subdivides a given simplicial complex. This can be considered as representing the complex with a finite set of weighted points. See Figure 1 as an example. Moreover, we also present a method to approximate an object with a union of balls via its subdividing alpha complex.

1.1 Motivation and Related Works

The motivation of this paper can be classified into two categories: the skin approximation and the conforming Delaunay complex.

Skin approximation. Our eventual goal is to approximate a given simplicial complex with the *skin surface*, which is a smooth surface based on a finite set of balls [7]. Amenta et. al. [1] have actually raised this question and the purpose was to perform deformation between polygonal objects. As noted in some previous works [2, 7], deformation can be performed robustly and efficiently with the skin surface. Our work here can be viewed as a stepping stone to our main goal.

As mentioned by Kruithof and Vegter [13], one of the first steps to approximate an object with a skin surface is to have a set of balls that approximate the object. For this purpose, we produce a set of balls whose alpha shape is the same

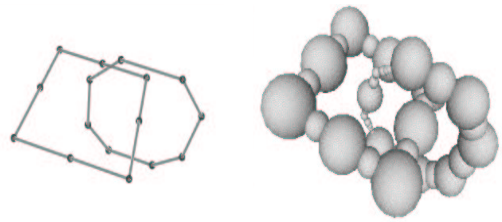

Fig. 1. An example of a subdividing alpha complex of a *link* embedded in $\mathbb{R}^3$. The right hand side of the figure shows a union of balls whose alpha complex resemble the input *link*, shown in the left hand side.

as the object. It is well known that such union of balls is homotopy equivalent to the object [6]. At the same time, we are able to produce a union of balls that approximate the object.

Approximating an object by a union of balls itself has applications in deformation. In such representation, shapes can be interpolated [15]. Some shape matching algorithms also use the union of balls representation [18]. We believe such approximation can also be useful for collision detection and coarse approximation [12].

Conforming Delaunay complex(CDC). The work on conforming Delaunay complex(CDC) are done mainly for the unweighted point set in two and three dimensional cases [3–5, 11, 14]. As far as our knowledge is concerned, there is no published work yet on the construction of CDC for any given simplicial complex in arbitrary dimension. The relation of CDC to our work here should be obvious, that is, we compute CDC of weighted points in arbitrary dimension.

1.2 Assumptions and Approach

The assumption of our algorithm is a constrained triangulation of a simplicial complex $\mathcal{C}$ is given, that is, a triangulation of the convex hull of $\mathcal{C}$ that contains $\mathcal{C}$ itself. An example of this is the constrained Delaunay triangulation of $\mathcal{C}$ [19, 20].

Our approach is to construct the subdividing alpha complex of the l-skeleton of $\mathcal{C}$ before its $(l+1)$-skeleton. For each simplex in the l-skeleton, we add weighted points until it is subdivided by the alpha complex. In the process, we maintain the invariant that the alpha shape is a subset of the underlying space of $\mathcal{C}$. This is done by avoiding two weighted points intersecting each other when their centers are not in the same simplex. For this purpose, we introduce the *protecting cells.*

The main issue in this approach is that we do not add infinitely many points, that is, our algorithm is able to terminate. To establish this, we guarantee that, for each simplex, there is a positive lower bound for the weight of the added point. This fact, together with the compactness of each simplex, ensures that only finitely many points are added into the simplex. We will formalize this fact in Section 5.

For our object approximation method, our main idea is to compute the subdividing alpha complex of the object by assigning very small weights to the points inserted in the boundary of the object. In this way, the weighted points in the interior will have relatively big weights and they make a good approximation for the object. Further clarification is in Section 6.

1.3 Outline

We start by describing some definitions and notations in Section 2. Then, in Section 3 we outline some properties which our algorithm aims at. Section 4 introduces the notion of *protecting cells*. Finally, we present our algorithm and method of object approximation in Sections 5 and 6, respectively. We conclude with certain remarks in Section 7.

2 Definitions and Notations

We briefly review some definitions and notations on simplicial and alpha complexes that we use in this paper.

Simplicial complexes. The convex hull of a set of points, $S \subseteq \mathbb{R}^d$, is denoted by $\mathrm{conv}(S)$. It is a *k-simplex* if $|S| = k+1$ and S is affinely independent. Let $\sigma = \mathrm{conv}(S)$ be a k-simplex, its *dimension* is denoted as $\dim(\sigma) = k$. For any $T \subset S$, $\tau = \mathrm{conv}(T)$ is also a simplex and it is called a *face* of σ, denoted by $\tau \prec \sigma$. We consider σ is not a face of itself. If $\dim(\tau) = l$, then τ is called an l-face of σ. Note that the faces of a simplex σ constitutes the boundary of σ, whereas, the interior of σ is σ minus all its faces. A *simplicial complex*, $\mathcal{C}$, is a set of simplices with the following properties.

1. If $\sigma \in \mathcal{C}$ and $\tau \prec \sigma$ then $\tau \in \mathcal{C}$, and,
2. If $\sigma_1, \sigma_2 \in \mathcal{C}$ then $\sigma_1 \cap \sigma_2$ is a face of both σ_1, σ_2.

The *underlying space* of $\mathcal{C}$ is the space occupied by $\mathcal{C}$, namely, $|\mathcal{C}| = \bigcup_{\sigma \in \mathcal{C}} \sigma$. We denote $\mathrm{vert}(\mathcal{C})$ to be the set of vertices in $\mathcal{C}$.

For a set of simplices $\mathcal{J} \subseteq \mathcal{C}$, the *closure* of $\mathcal{J}$, denoted by $\mathrm{cl}(\mathcal{J})$, is the minimal simplicial complex that contains $\mathcal{J}$. For $\sigma \in \mathcal{C}$, the *star* of σ, denoted by $\mathrm{star}(\sigma)$, is the set of simplices in $\mathcal{C}$ that contains σ.

A *constrained triangulation* T_c of a simplicial complex $\mathcal{C}$ is a triangulation of the convex hull of $|\mathcal{C}|$ in which $\mathcal{C} \subseteq T_c$. Note that not all simplicial complexes can be triangulated without additional vertices, e.g. the Schönhardt polyhedron [16].

Alpha complexes. A *ball* $b = (z, w) \in \mathbb{R}^d \times \mathbb{R}$, is the set of points whose distance to z is less than or equal to $\sqrt{w}$. We also call this a *weighted point* with center z and weight w. The *weighted distance* of a point $p \in \mathbb{R}^d$ to a ball $b = (z, w)$ is defined as $\pi_b(p) = \|pz\|^2 - w$. $\bigcup X$ is to denote the union of a set of balls X and $\bigcap X$ is to denote the common intersection of X.

Let B be a finite set of balls in $\mathbb{R}^d$. The *Voronoi cell* of a ball $b \in B$, ν_b, is the set of points in $\mathbb{R}^d$ whose weighted distance to b is less than or equal to any other ball in B. For $X \subseteq B$, the *Voronoi cell* of X is $\nu_X = \bigcap_{b \in X} \nu_b$. The *Voronoi complex* of B is $V_B = \{\nu_X \mid X \subseteq B \text{ and } \nu_X \neq \emptyset\}$.

For a ball b and a set of balls X, we denote $z(b)$ to be the center of b and $z(X)$ to be the ball centers of X. The Delaunay complex of B is the collection of simplices,

$$D_B = \{\delta_X = \operatorname{conv}(z(X)) \mid \nu_X \in V_B\}.$$

The *alpha complex* of B is a subset of the Delaunay complex D_B which is defined as follow [6],

$$\mathcal{K}_B = \{\delta_X \mid \bigcup X \cap \nu_X \neq \emptyset, \nu_X \in V_B\}.$$

The *alpha shape* of B is the underlying space of $\mathcal{K}_B$, namely, $|\mathcal{K}_B|$. Remark that if $\delta_X \in \mathcal{K}_B$ then $\bigcap X \neq \emptyset$.

3 Conditions for Subdividing Alpha Complex

The alpha complex of a set of balls, B, is said to *subdivide* a simplicial complex, $\mathcal{C}$, if (i) $|\mathcal{K}_B| = |\mathcal{C}|$, and (ii) every simplex in $\mathcal{K}_B$ is contained in a simplex in $\mathcal{C}$. We have the following main theorem that is used to construct the subdividing alpha complex.

Theorem 1. *Let B be a set of balls and $\mathcal{C}$ be a simplicial complex. If B satisfies the following Conditions C1 and C2:*

C1. *for a subset $X \subseteq B$, if $\bigcap X \neq \emptyset$ then $z(X) \subseteq \sigma$ for some $\sigma \in \mathcal{C}$, and,*
C2. *for each $\sigma \in \mathcal{C}$, define $B(\sigma) = \{b \in B \mid b \cap \sigma \neq \emptyset\}$.*
Then we have: $z(B(\sigma)) \subseteq \sigma \subseteq \bigcup B(\sigma)$,

then $\mathcal{K}_B$ subdivides $\mathcal{C}$.

The proof is fairly tedious and it is presented in the next subsection.

Condition C1 states that for a set of balls whose common intersection is not empty, their centers must be in the same simplex in $\mathcal{C}$. This also implies that the center of each ball must be in $\mathcal{C}$. In Condition C2, we require that if a ball intersects with a simplex then its center must be in the simplex. Furthermore, the simplex is covered by a set of balls whose centers are in the simplex.

To construct a set of balls that satisfies Condition C1, we introduce the notion of *protecting cell* for each simplex in $\mathcal{C}$, which is defined by the barycentric subdivision. We use the protecting cell in order to control the weight of each point. This will be discussed in Section 4. We show how to achieve Condition C2 in Section 5. Figure 1 on the front page illustrates an example of a subdividing alpha complex satisfying Conditions C1 and C2.

3.1 Proof of Theorem 1

It is obvious that the following two properties are equivalent to the criteria for subdividing alpha complex:

P1. Every simplex in $\mathcal{K}_B$ is contained in a simplex in $\mathcal{C}$.
P2. Every simplex in $\mathcal{C}$ is a union of some simplices in $\mathcal{K}_B$.

We divide the theorem into two lemmas. Lemma 1 states that Condition C1 implies property P1, while Lemma 2 states that Condition C2 implies property P2.

Lemma 1. *If B satisfies Condition C1, then every simplex in $\mathcal{K}_B$ is contained in a simplex in $\mathcal{C}$, that is, property P1.*

Proof. It is immediate that every vertex in $\mathcal{K}_B$ is inside a simplex in $\mathcal{C}$. Let δ_X be a simplex in $\mathcal{K}_B$. By the remark in the definition of alpha complex, $\bigcap X \neq \emptyset$. Then, by Condition C1, all their centers are in the same simplex $\sigma \in \mathcal{C}$. Therefore, $\delta_X = \mathrm{conv}(z(X)) \subseteq \sigma$. □

Lemma 2. *If B satisfies Condition C2, then every simplex in $\mathcal{C}$ is a union of some simplices in $\mathcal{K}_B$, that is, property P2.*

Proof. Let $\sigma \in \mathcal{C}$, recall that $B(\sigma) \subseteq B$ is the set of balls which intersect with σ and each of their centers are inside σ. We consider $D_{B(\sigma)}$, the Delaunay complex of the balls $B(\sigma)$. To prove σ is a union of simplices in $\mathcal{K}_B$, we prove the following claims:

A. σ is a union of simplices in $D_{B(\sigma)}$, and,
B. every simplex in $D_{B(\sigma)}$ is a simplex in $\mathcal{K}_B$.

These two claims establish Lemma 2.

The proof of Claim A is as follow. For any vertex of σ, there exists a ball $b \in B(\sigma)$ centered on σ. Furthermore, the centers of balls of $B(\sigma)$ are inside σ. Thus, σ is the convex hull of $z(B(\sigma))$. It is a fact that the convex hull of a set of points is the union of its Delaunay simplices. Therefore, σ is a union of simplices in $D_{B(\sigma)}$.

For Claim B, we first show that *if* $\delta_X \in \mathrm{D}_{B(\sigma)}$ *then* $\nu_X \cap \sigma \neq \emptyset$, where $\nu_X \in V_{B(\sigma)}$. The intuitive meaning is that every Voronoi cell in $V_{B(\sigma)}$ always intersects σ. We prove it by induction on $\dim(\sigma)$.

The base case is $\dim(\sigma) = 0$. It is true by Condition C2. Assume the statement is true for every $k-1$-simplex in $\mathcal{C}$.

Let $\dim(\sigma) = k$ and δ_X be a simplex in $D_{B(\sigma)}$. There are two cases:

1. $X \subseteq B(\tau)$, where $\tau \prec \sigma$.
Consider the Voronoi cell $\nu'_X \in V_{B(\tau)}$ and the Voronoi cell $\nu_X \in V_{B(\sigma)}$. Under Condition C2, each point in τ has negative distance to some ball in $B(\tau)$ and positive distance to every ball in $B(\sigma) - B(\tau)$. This means the Voronoi cell ν'_X is not effected by the additional balls $B(\sigma) - B(\tau)$, that is, $\nu_X \cap \tau = \nu'_X \cap \tau$. Applying the induction hypothesis, $\nu_X \cap \tau = \nu'_X \cap \tau \neq \emptyset$. In particular, since $\tau \prec \sigma$, we have $\nu_X \cap \sigma \neq \emptyset$.
2. $X \not\subseteq B(\tau)$, for any $\tau \prec \sigma$.
This means X contain some balls which do not belong to $B(\tau)$, for any $\tau \prec \sigma$. Let b be such a ball, that is, $b \in X - \{b' \mid b' \in B(\tau), \tau \prec \sigma\}$. By Condition C2, each point in τ has negative distance to some balls in $B(\tau)$ and positive distance to b. Since Voronoi cell ν_b is convex and $\nu_b \neq \emptyset$, we have the Voronoi cell of a ball $b \in B(\sigma)$ lies entirely in the interior of σ. In particular, the Voronoi cell ν_X is inside the interior of σ. Therefore, $\nu_X \cap \sigma \neq \emptyset$.

Thus, this proves that for every $\delta_X \in D_{B(\sigma)}$, $\nu_X \cap \sigma \neq \emptyset$ where $\nu_X \in V_{B(\sigma)}$.

Back to Claim B, we prove that if a simplex δ_X belongs to $D_{B(\sigma)}$, then $\delta_X \in \mathcal{K}_B$. Note that by result above, if $\delta_X \in D_{B(\sigma)}$ then $\nu_X \cap \sigma \neq \emptyset$ where $\nu_X \in V_{B(\sigma)}$. Under Condition C2, σ is covered $\bigcup B(\sigma)$. Thus, for every point $p \in \nu_X \cap \sigma$, p has negative distance to some balls in $B(\sigma)$, in particular, p has negative distance to all balls in X. Therefore, $\bigcup X \cap \nu_X \neq \emptyset$. Moreover, by Condition C2, p has positive distance to every ball which does not belong to $B(\sigma)$. This implies that p still belongs to the Voronoi cell of X in V_B, thus, δ_X is also a simplex in D_B. This proves that $\delta_X \in \mathcal{K}_B$. □

4 Achieving Condition C1

We divide this section into two subsections. In Subsection 4.1 we give a formal but brief construction of barycentric subdivision of a simplicial complex. In Subsection 4.2 we define our notion of protecting cells. For some discussions of barycentric subdivision, we refer the reader to [17].

4.1 Barycentric Subdivision

Let σ be a k-simplex with vertices $S = \{s_1, \dots, s_{k+1}\}$. The barycenter of σ is denoted by $\overline{\sigma}$, or $\overline{S} = \frac{1}{k+1} \sum_{i=1}^{k+1} s_i$.

Definition 1. *Let $\sigma = \mathrm{conv}(S)$ be a simplex and $T \subseteq S$. For any $t \in T$, denote $\sigma_T(t) = \mathrm{conv}(S \cup \{\overline{T}\} - \{t\})$. The subdivision of σ by the barycenter of* $\mathrm{conv}(T)$ *is the set of simplices:* $\mathrm{subdiv}(\sigma, T) = \{\sigma_T(t) \mid t \in T\}$.

Let $\mathcal{C}$ be a simplicial complex embedded in $\mathbb{R}^d$. We have a sequence of complexes $\mathcal{C}^0, \mathcal{C}^1, \dots, \mathcal{C}^d$ which is defined inductively as follows:

Definition 2. *Let $\mathcal{C}^0$ be a constrained triangulation of $\mathcal{C}$. The simplicial complex* $\mathcal{C}^j = \mathrm{cl}(\{\mathrm{subdiv}(\sigma, T) \mid \sigma \in \mathcal{C}^{j-1}\})$ *where*

1. $\sigma = \mathrm{conv}(S)$ *is of dimension* d*, and,*
2. $T = S \cap \mathrm{vert}(\mathcal{C}^0)$.

The simplicial complex $\mathcal{C}^d$ is called *the barycentric subdivision* of $\mathcal{C}^0$. We have the following fact concerning $\mathcal{C}^0$ and $\mathcal{C}^d$.

Fact 1. *There is a 1-1 correspondence between simplices in* $\mathcal{C}^0$ *and vertices in* $\mathcal{C}^d$*. More precisely, each simplex in* $\mathcal{C}^0$ *corresponds to its barycenter in* $\mathcal{C}^d$*.*

4.2 Protecting Cells

Given a simplicial complex $\mathcal{C}$ in $\mathbb{R}^d$, let $\mathcal{C}^d$ be the barycentric subdivision of $T_{\mathcal{C}}$, a constrained triangulation of $\mathcal{C}$. We use Fact 1 to define the protecting cells of simplices in $\mathcal{C}$.

Definition 3. *Let* $\sigma \in \mathcal{C}$*. The* protecting cell *of* σ*, denoted by* ψ_σ*, is defined as the closure of the star of the barycenter of* σ *in* $\mathcal{C}^d$*, namely,*

$$\psi_\sigma = \mathrm{cl}(\mathrm{star}(\overline{\sigma})),$$

where $\overline{\sigma}$ *is a vertex in* $\mathcal{C}^d$*.*

Figure 2 illustrates parts of protecting cells of various simplices in $\mathbb{R}^2$.

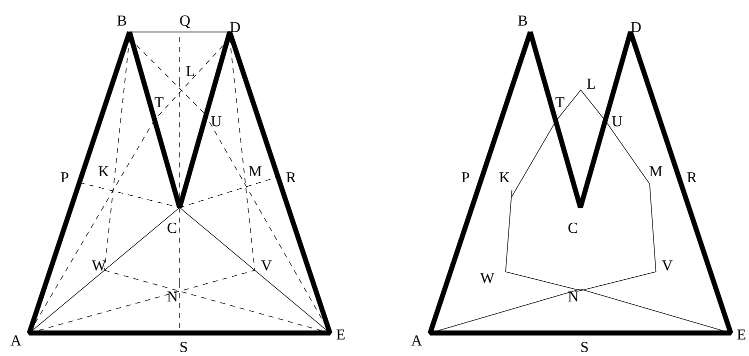

Fig. 2. Suppose we have the polygon $ABCDE$ as simplicial complex $\mathcal{C}$ embedded in $\mathbb{R}^2$. The left figure illustrates the barycentric subdivision of $T_{\mathcal{C}}$. Vertices P, Q, R, S, T, U, V, W are barycenters of the edges in $T_{\mathcal{C}}$. Vertices K, L, M, N are barycenters of the triangles in $T_{\mathcal{C}}$. The right figure shows the protecting cells of the vertex C and the edge AE, respectively. ψ_C is the polygon $KTLUMVNW$, while ψ_{AE} is the triangle AEN.

Let p be a point in the interior of σ. Consider the *link* of the centroid of σ, $\overline{\sigma}$, in $\mathcal{C}^d$, that is, $\mathrm{cl}(\mathrm{star}(\sigma)) - \mathrm{star}(\sigma)$. This link uniquely defines the maximal ball with the center on p and not intersecting any simplex which is not in ψ_σ. We denote the weight of such maximal ball by $\mathrm{MaxWeight}(p)$. We call $\mathrm{MaxWeight}(p)$ the *maximum weight* of p. The value $\mathrm{MaxWeight}(p)$ can be computed by finding the distance from p to the nearest bounding $(d-1)$-simplices.

Proposition 1. *Let $p_1, p_2 \in |\mathcal{C}|$. Suppose p_1 is in the interior of $\sigma_1 \in \mathcal{C}$ and p_2 is in the interior of $\sigma_2 \in \mathcal{C}$. If σ_1 and σ_2 are not faces of each other then the two balls $(p_1, \gamma \cdot \mathrm{MaxWeight}(p_1))$ and $(p_2, \gamma \cdot \mathrm{MaxWeight}(p_2))$ do not intersect for any $\gamma < 1$.*

Proof. We observe that if σ_1 and σ_2 are not faces of each other then their protecting cells can only intersect in their boundary. Thus, the proposition follows.

Therefore, Condition C1 can be achieved if all balls in B have their weight strictly less than the maximum weight of their centers.

5 Algorithm

The input is a simplicial complex $\mathcal{C}$ embedded in $\mathbb{R}^d$, together with its triangulation T_c. As stated in Subsection 1.2, our algorithm will subdivide the l-skeleton of $\mathcal{C}$, starting from $l = 0$ up to $l = d$. For each simplex σ in $\mathcal{C}$, we will construct the set of balls $B(\sigma)$ by executing the procedure `ConstructBalls`(σ). (Recall the definition of $B(\sigma)$ as stated in Theorem 1.)

Before we proceed to describe the details of `ConstructBalls`(σ) in the next subsection, we need the following concept of restricted Voronoi complex.

For a set of balls $B \subset \mathbb{R}^d \times \mathbb{R}$, consider the restriction of V_B on a k-simplex $\sigma \in \mathcal{C}$. The *restricted Voronoi cell* of $X \subseteq B$ is $\nu_X(\sigma) = \nu_X \cap \sigma$. Similarly, the *restricted Voronoi complex* $V_B(\sigma)$ is the collection of the restricted Voronoi cells. For convenience, we also include the intersection of $\nu_X(\sigma)$ with faces of σ into $V_B(\sigma)$. That is, $V_B(\sigma) = \{\nu_X(\tau) \mid \tau \in \mathrm{cl}(\sigma)\}$.

Let $\sigma \in \mathcal{C}$ be an k-simplex. For a set of balls X, consider its restricted Voronoi complex on σ, $V_X(\sigma)$. We define the following terms that will be used in this subsection. A Voronoi vertex v in $V_X(\sigma)$ is called a *negative*, *zero* or *positive* vertex, if $\pi_b(v) < 0$, $\pi_b(v) = 0$, or $\pi_b(v) > 0$, respectively, where v is the Voronoi vertex in the Voronoi cell of $b \in X$, i.e. $v \in \nu_b(\sigma)$. Note that if a vertex is positive then it is outside every ball in X.

5.1 Procedure `ConstructBalls`(σ)

Procedure 1 describes the details of `ConstructBalls()` . In the procedure, we denote γ by a real constant where $0 < \gamma < 1$. Recall also that a ball centered at a point u with weight w is written as (u, w).

It is obvious that the whole algorithm produces a correct set of balls B provided that the procedure `ConstructBalls`(σ) produces the correct balls $B(\sigma)$ for each $\sigma \in \mathcal{C}$. Since the weights of constructed balls are all strictly less than the maximum weights of the centers, Condition C1 is achieved by Proposition 1. The following Proposition 2 ensures that Condition C2 is achieved provided that the procedure `ConstructBalls()` terminates. We establish the termination of our algorithm in Theorem 2.

Proposition 2. *Let X be a set of balls. Suppose $z(X) \subseteq \sigma$. Then $\sigma \subseteq \bigcup X$ if and only if there is no positive vertex in $\mathrm{V}_X(\sigma)$.*

Procedure 1 `ConstructBalls`(σ)

if dim $\sigma = 0$ **then**
  $B(\sigma) := (\sigma, \gamma \cdot \mathrm{MaxWeight}(\sigma))$
else
  Let $l :=$ dim σ
  Let $\tau_1, \ldots, \tau_{l+1}$ be the $(l-1)$-faces of σ.
  $X := B(\tau_1) \cup \cdots \cup B(\tau_{l+1})$
  while $\exists$ a positive vertex u in $V_X(\sigma)$ **do**
    $w := \gamma \cdot \mathrm{MaxWeight}(u)$
    $X := X \cup \{(u, w)\}$
  end while
  $B(\sigma) := X$
end if

Proof. ($\Rightarrow$) Suppose X covers σ. Let v be an arbitrary Voronoi vertex of $\nu_b(\sigma)$ for some ball $b \in X$. If $\pi_b(v) > 0$ then for any $b' \in X$, $\pi_{b'}(v) \geq \pi_b(v) > 0$, thus, contradicts our assumption that $\sigma \subseteq \bigcup X$. Therefore, every voronoi vertex is not a positive vertex.

($\Leftarrow$) Suppose there is no positive Voronoi vertex in $V_X(\sigma)$. We claim that $\nu_b(\sigma) \subseteq b$ for all $b \in X$. This claim follows from the fact that $\nu_b(\sigma)$ is bounded and is indeed the convex hull of its Voronoi vertices. So, by our assumption that the Voronoi vertices are not positive, it is immediate that $\nu_b(\sigma) \subseteq b$ for any $b \in X$. Since σ is partitioned into $\nu_b(\sigma)$ for all $b \in X$, it follows that $\sigma \subseteq \bigcup X$.

Theorem 2. *The procedure* `ConstructBalls`(σ) *terminates for any $\sigma \in \mathcal{C}$ and each weighted point in $B(\sigma)$ has positive weight.*

5.2 Proof of Theorem 2

The proof of Theorem 2 is based on the following proposition and the fact that each simplex is compact.

Proposition 3. *Let Λ be a subset of σ whose boundary lies entirely in the interior of σ. Then there exists a constant $c > 0$ such that for all $p \in \Lambda$,* $\mathrm{MaxWeight}(p) > c$.

Proof. For a point p in the interior of a simplex $\sigma \in \mathcal{C}$, $\mathrm{MaxWeight}(p) > 0$, since it has nonzero distance to all other faces of d-simplex in ψ_σ. Let $p_1, p_2, \ldots$ be a convergent sequence of points in σ. Suppose $\{p_i\}$ converges to p. $\mathrm{MaxWeight}(\cdot)$ is a continuous function. So, $\lim_{i\to\infty} \mathrm{MaxWeight}(p_i) = \mathrm{MaxWeight}(\lim_{i\to\infty} p_i) = \mathrm{MaxWeight}(p) = 0$ if and only if p is in the boundary of σ. Since the boundary of Λ lies entirely inside the interior of σ, the infimum of the set $\mathrm{MaxWeight}(\Lambda) > 0$. Thus, our proposition follows.

The fact that each simplex $\sigma \in \mathcal{C}$ is compact can be rephrased as follow: For every sequence of points $p_1, p_2, \ldots, p_n, \ldots$, where each $p_i \in \sigma$, there exists a

Cauchy subsequence that converges to a point $p \in \sigma$. For the detail discussion we refer the reader to [17].

We prove Theorem 2 by induction on $\dim(\sigma)$. The base case is $\dim(\sigma) = 0$. It is immediate that the procedure terminates and its weight is greater than zero. Assume that for any simplex of dimension $k-1$ Theorem 2 holds.

Let $\dim(\sigma) = k$. We apply the induction hypothesis on the $(k-1)$-faces of σ. Let $\{\tau_1, \ldots, \tau_{k+1}\}$ be the $(k-1)$-faces of σ. By induction hypothesis, the procedure `ConstructBalls`(τ_i) terminates for each τ_i and balls in $B(\tau_i)$ have weights greater than zero. Consider the space $\Lambda = \sigma - \bigcup_{1 \leq i \leq k+1} B(\tau_i)$. Since all balls in each $B(\tau_i)$ have weights greater than zero, the boundary of Λ lies entirely inside interior of σ. By Proposition 3, there exists a constant $c > 0$ such that MaxWeight of each point in this space is greater than c.

Assume the contrary that `ConstructBalls`(σ) does not terminate. Thus, it inserts infinitely many balls into X, (u_i, w_i) for $i = 0, 1, 2, \ldots$ with (u_i, w_i) is inserted first before (u_{i-1}, w_{i-1}). According to the procedure, the balls are inserted at a positive vertex, thus, each u_i is not inside the ball (u_j, w_j) with $i > j$.

Since σ is compact there exists a Cauchy subsequence of centers of the balls $u_{k_1}, \ldots, u_{k_n}, \ldots$ with $k_i \geq i$. We apply the Cauchy sequence criteria with some $\epsilon < \gamma \cdot c$. Thus, there exists N such that $|u_{k_i} u_{k_j}| < \epsilon < c$ for any $i, j > N$. Assume $k_i < k_j$, this means u_{k_j} is inside the ball (u_{k_j}, w_{k_j}) since $w_{k_j} > c$. This contradicts that u_{k_j} is a positive vertex. Therefore, the procedure `ConstructBalls`(σ) terminates.

6 Object Approximation

Let $\mathcal{O}$ be a simplicial complex representing an object in $\mathbb{R}^3$ and $\mathcal{C}$ be its boundary such that $|\mathcal{C}|$ is a piecewise linear 2-manifold. For a given real positive number ϵ, we can construct a subdividing alpha complex of $\mathcal{C}$ such that the weighted points produced have weights less than ϵ. We achieve this by the following modification. Replace line 8 in procedure `ConstructBalls`(σ) with the instruction below:

$$\textbf{if } (\gamma \cdot \text{MaxWeight}(u) > \epsilon) \textbf{ then } w := \epsilon \textbf{ else } w := \gamma \cdot \text{MaxWeight}(u)$$

It should be obvious that our algorithm is still correct and able to terminate.

Let $\Delta = \{\delta_X \in D_B \mid D_X \subseteq |\mathcal{O}|\}$, that is, all the Delaunay tetrahedra that are inside $\mathcal{O}$. Each Delaunay tetrahedron determines a sphere which is orthogonal to all the four weighted points. We consider the collection of all such balls B' and observe that $\bigcup B'$ makes a good approximation of $\mathcal{O}$. We make the following observations:

1. All balls in B' have positive weights and does not intersect with $\mathcal{C}$.
2. The space $\mathcal{O} - \bigcup B$ is fully covered by the balls in B'.
3. The boundary of $\bigcup B'$ is homeomorphic to $|\mathcal{C}|$.
4. The Hausdorff distance from $\mathcal{O}$ to $\bigcup_{b \in B'} b'$ is less than ϵ.

7 Concluding Remarks

In this paper we propose an algorithm to compute an alpha complex that subdivides a simplicial complex. We also show how via subdividing alpha complex we can approximate a closed polygonal object. It should be obvious that the approximation method can be generalized fairly easily to arbitrary dimension.

Discussion. The subdividing alpha complex discussed here is the weighted alpha complex. Figure 3 shows that a simple example where unweighted subdividing alpha complex is not always possible.

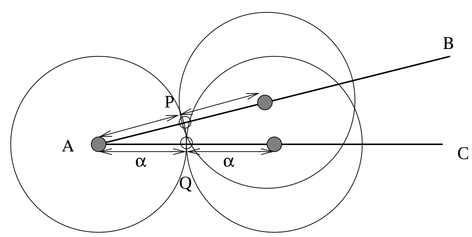

Fig. 3. The unweighted subdividing alpha cannot exist when $\angle A < 2\arcsin(\frac{1}{4})$. There must exist a ball centered on A. Also, there must be some balls centered on the segments $\overline{AP}$ and $\overline{AQ}$. Those balls will inevitably intersect and create an extra edge in the alpha complex.

One point worth mentioning here is that the number of balls needed for subdividing alpha complex does not depend on the combinatorial properties of the given $\mathcal{C}$. Figure 4 illustrates a relatively simple simplicial complex which requires huge number of balls for its subdividing alpha complex.

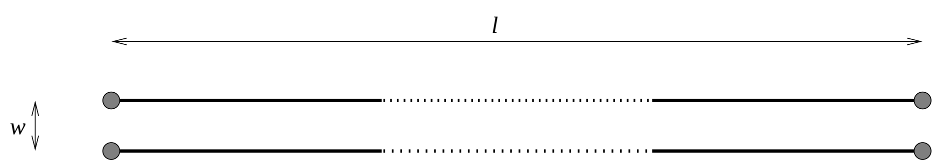

Fig. 4. The simplicial complex consists of only four vertices and two parallel edges. The number of weighted points needed for subdividing alpha complex will be greater than $\frac{2\times l}{w}$. So if l is much bigger than w then the number of weighted points needed can be huge too.

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