Simplifying Alpha Complexes

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Abstract. Given a set of weighted points $B \subseteq \mathbb{R}^2 \times \mathbb{R}$, we show an algorithm to construct another set of weighted points $\widehat{B}$ with fewer disks and the same alpha shape.

1 Introduction

Given a set of weighted points, or disks, $B \subseteq \mathbb{R}^2 \times \mathbb{R}$, one can compute the alpha complex as defined by Edelsbrunner et. al. [1, 2]. Intuitively, the alpha complex of a set of disks, $\mathcal{K}_B$, is a subcomplex of the Delaunay complex of B whose underlying space is homotopy equivalent to the union of disks, $\bigcup B$. The underlying space of an alpha complex is called the alpha shape, denoted as $|\mathcal{K}_B|$.

In this paper, we consider the following problem. Given a set of disks B, we want to construct another set of disks $\widehat{B}$ with less disks than B and $|\mathcal{K}_B| = |\mathcal{K}_{\widehat{B}}|$. The motivation of the problem is to simplify the alpha complex produced by the algorithm in [3]. The simplification is necessary because the resulting subdividing alpha complex of a simplicial complex may end up in huge number of disks.

Our approach to the problem is to increase the radius of each disk in B, hoping that some other disks will become redundant. We define the following step for our abstract algorithm. For each disk $b \in B$, increase its weight until it causes a flip that changes the Delaunay complex. Repeat the step until there is a change in the alpha shape. When it happens, our algorithm undoes the execution of the last step and keeps the alpha shape unchanged. Although our work here is only on the two dimensional space, it will be obvious that it can be extended into arbitrary dimensional space. Another related work is in [4, 5] where the algorithm to compute dynamically the Delaunay triangulation by deletion of a vertex is proposed.

1.1 Background

We briefly review the definition of alpha complex. Let $B \subseteq \mathbb{R}^2 \times \mathbb{R}$ be a set of disks (weighted points). The weighted distance of a point p to a disk b with center z and radius r is $|zp|^2 - r^2$. The Voronoi cell of $b \in B$, denoted by ν_b, is the pointset with the least weighted distance to b. Let $X \subseteq B$ and suppose that the Voronoi cell of X, $\nu_X = \cap_{b \in X} \nu_b$ is not empty. Its corresponding Delaunay simplex δ_X, which is the convex hull of the centers in X, is an *alpha* simplex if $\bigcup X \cap \nu_X \neq \emptyset$. Otherwise, we call δ_X a non-alpha simplex. The alpha complex of B is the complex consists of all alpha simplices, denoted as K_B.

The underlying space of an alpha complex is called the *alpha shape*, denoted by $|\mathcal{K}_B|$. It can be viewed as a collection of polygons, line segments and vertices. Furthermore, the intersection of any two of them[1] is either empty or a set consists of only one point.

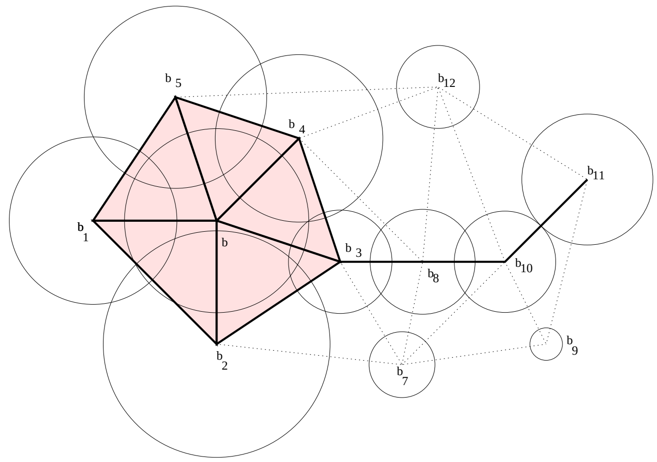

Fig. 1. The alpha complex is a subset of the Delaunay complex of the weighted points $\{b_1, \ldots, b_{12}\}$. The alpha shape consists of one polygon $b_1b_2b_3b_4b_5$, two line segments b_3b_{10} and $b_{10}b_{11}$ and three vertices b_7, b_9 and b_{12}.

2 Description of the Algorithm

Our algorithm increases the weight of each disk of the given set B, while at the same time maintaining the following two invariants:

1. $|\mathcal{K}_B| \subseteq |\mathcal{K}_{B'}|$, and,
2. $|\mathcal{K}_{B'}| \subseteq |\mathcal{K}_B|$.

where B' denotes the set of disks with the increased radii. In the next two subsections, we will describe how to maintain those two invariants.

2.1 Maintaining $|\mathcal{K}_B| \subseteq |\mathcal{K}_{B'}|$.

It may not seem obvious but the alpha shape, $|\mathcal{K}_B|$, may shrink if we increase the radius of a disk in B. For example, recall that a Delaunay simplex δ_X belongs to the alpha complex $\mathcal{K}_B$ if the union of disks in X, namely $\bigcup X$, intersects with

[1] The intersection can be a polygon with another polygon, a line segment with another line segment, or a polygon with a line segment.

the Voronoi cell ν_X in the Voronoi complex of B [1]. By increasing the radius, we may push the Voronoi cell ν_X away and out of $\bigcup X$, thus, the simplex δ_X does not belong to the alpha complex anymore.

In order to keep the Voronoi vertex of a triangle within the union of disks, we limit the expansion of any disks by the *corner points*. The corner points, C_B, are the points on the boundary of the union of disks which is the intersection of more than one disks. Then, when we increase the radius of a disk, b, we make sure that it will not cover any points in $C_B - \partial b$, in which ∂b denotes the boundary of b.

For a line segment in $\mathcal{K}_B$, the same problem may occur. Let b_1 and b_2 be the disks with centers at the two endpoints of the line segment. It will disappear when we increase the radius of one disk within the line segment such that it contains either b_1 or b_2. Thus, we just keep growing some disks without containing b_1 and b_2.

2.2 Maintaining $|\mathcal{K}_{B'}| \subseteq |\mathcal{K}_B|$.

Our idea is to preserve every non-alpha simplex. To do this, we only consider the disks whose stars contain some non-alpha simplices.

Preserving the non-alphaness of an edge. There are two possible reasons why a Delaunay edge $\delta_{\{b_1,b_2\}}$ is non-alpha.

1. $b_1 \cap b_2 = \emptyset$, or,
2. there exist a disk b such that $\delta_{\{b,b_1\}}$ and $\delta_{\{b,b_2\}}$ are alpha edges and b covers both corner points of $\delta_{\{b_1,b_2\}}$.

To maintain the first case, we increase the radius of b_1 without touching b_2. For the second case, the disk b_1 must not cover any corner point of $\delta_{\{b,b_2\}}$. Similarly, the disk b_2 must not cover any corner point of $\delta_{\{b,b_1\}}$.

Preserving the non-alphaness of a triangle. In this case, we can assume that all edges of the triangles are alpha edges. The reason is that if the non-alphaness of the edges is maintained then it makes a non-alpha triangle remains non-alpha.

Let the edges $\delta_{\{b_1,b_2\}}$, $\delta_{\{b_2,b_3\}}$ and $\delta_{\{b_3,b_1\}}$ are alpha edges, whereas, $\delta_{\{b_1,b_2,b_3\}}$ is a non-alpha triangle. The maintenance of the non-alphaness of $\delta_{\{b_1,b_2,b_3\}}$ follows if the radius of b_1 is increased such that it does not cover any corner point $C_B - \partial b_1$.

3 Conclusion

We presented an algorithm to decrease the number of weighted points without changing its alpha shape. About the complexity, it is easy to see that the corner points can be computed directly once the Delaunay complex has been constructed. Furthermore, when we increase the weight of a disk we can directly modify all corner points of the simplices surrounding the disk. In the two dimensional cases the number of simplices is linearly proportional to the number of vertices. Therefore, the number of times we do modification of corner points is linearly proportional to the number of simplices. Thus, the complexity of our algorithm is the same as the complexity of computing the initial Delaunay complex.

References

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