Uncertainty and Bayesian Networks

Chapters 13 and 14

Last Time

Outline

- Uncertainty
  - Probability
  - Syntax and Semantics
  - Inference
  - Independence and Bayes' Rule
- Bayesian Networks
  - Syntax and Semantics

Uncertainty

Let action A_t = leave for airport $_t$ minutes before flight
Will A_t get me there on time?

Problems:

1. partial observability (road state, other drivers' plans, etc.)
2. noisy sensors (traffic reports)
3. uncertainty in action outcomes (flat tire, etc.)
4. immense complexity of modeling and predicting traffic

Hence a purely logical approach either

1. risks falsehood: "A_{25} will get me there on time", or
2. leads to conclusions that are too weak for decision making:

"A_{25} will get me there on time if there's no accident on the bridge and it doesn't rain and my tires remain intact etc etc."

(A_{1440} might reasonably be said to get me there on time but I'd have to stay overnight in the airport …)

Methods for handling uncertainty

- Default or nonmonotonic logic:
  - Assume my car does not have a flat tire
  - Assume A_{25} works unless contradicted by evidence
- Issues: What assumptions are reasonable? How to handle contradiction?

- Rules with fudge factors:
  - A_{25} $\mapsto_{0.3}$ get there on time
  - *Sprinkler* $\mapsto_{0.99}$ *WetGrass*
  - *WetGrass* $\mapsto_{0.7}$ *Rain*
- Issues: Problems with combination, e.g., *Sprinkler* causes *Rain*??

- Probability
  - Model agent's degree of belief
  - Given the available evidence,
  - A_{25} will get me there on time with probability 0.04

Probability

Probabilistic assertions summarize effects of

- laziness: failure to enumerate exceptions, qualifications, etc.
- ignorance: lack necessary knowledge, initial conditions, etc.
- ignorance: lack of relevant facts, initial conditions, etc.

Subjective probability:

- Probabilities relate propositions to agent's own state of knowledge

  e.g., $P(A_{25} \mid \text{no reported accidents}) = 0.06$

These are not assertions about the world

Probabilities of propositions change with new evidence:

e.g., $P(A_{25} \mid \text{no reported accidents, 5 a.m.}) = 0.15$

Making decisions under uncertainty

Suppose I believe the following:

$P(A_{25}$ gets me there on time | …) = 0.04
$P(A_{90}$ gets me there on time | …) = 0.70
$P(A_{120}$ gets me there on time | …) = 0.95
$P(A_{1440}$ gets me there on time | …) = 0.9999

- Which action should the agent choose?

  Depends on its preferences for missing flight vs. time spent waiting, etc.
  - Utility theory is used to represent and infer preferences
  - Decision theory = probability theory + utility theory

Syntax

- Basic element: random variable
- Similar to propositional logic: possible worlds defined by assignment of values to random variables.
- Boolean random variables
  e.g., *Cavity* (do I have a cavity?)
- Discrete random variables
  e.g., *Weather* is one of <*sunny,rainy,cloudy,snow*>
- Domain values must be exhaustive and mutually exclusive
- Elementary proposition constructed by assignment of a value to a random variable: e.g., *Weather = sunny*, *Cavity = false*
  (abbreviated as ¬cavity)
- Complex propositions formed from elementary propositions and standard logical connectives e.g., *Weather = sunny* ∨ *Cavity = false*

Syntax

- Atomic event: A complete specification of the state of the world about which the agent is uncertain

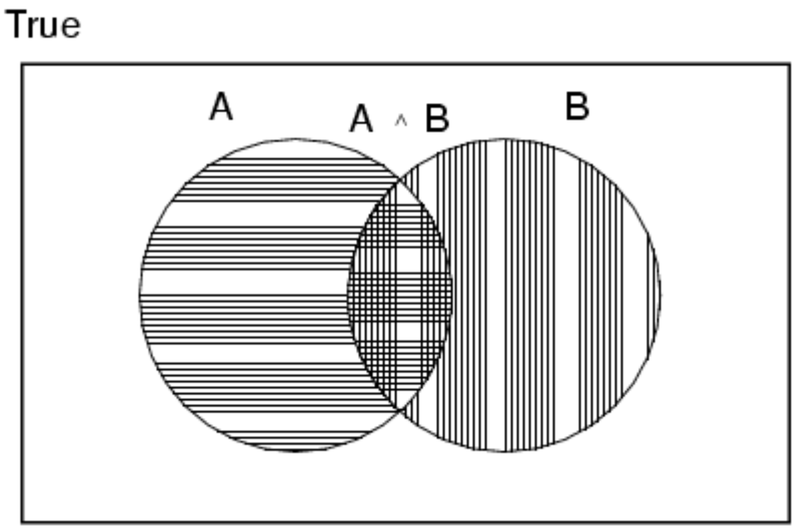
  E.g., if the world consists of only two Boolean variables *Cavity* and *Toothache*, then there are 4 distinct atomic events:

  $Cavity = false \wedge Toothache = false$
  $Cavity = false \wedge Toothache = true$
  $Cavity = true \wedge Toothache = false$
  $Cavity = true \wedge Toothache = true$

- Atomic events are mutually exclusive and exhaustive

Axioms of probability

- For any propositions *A*, *B*
  - $0 \leq P(A) \leq 1$
  - $P(true) = 1$ and $P(false) = 0$
  - $P(A \vee B) = P(A) + P(B) - P(A \wedge B)$



Prior probability

- Prior or unconditional probabilities of propositions
  e.g., P(*Cavity* = true) = 0.1 and P(*Weather* = sunny) = 0.72 correspond to belief prior to arrival of any (new) evidence

- Probability distribution gives values for all possible assignments:
  P(*Weather*) = <0.72,0.1,0.08,0.1> (normalized, i.e., sums to 1)

- Joint probability distribution for a set of random variables gives the probability of every atomic event on those random variables
  P(*Weather*,*Cavity*) = a 4 × 2 matrix of values:

Weather =	sunny	rainy	cloudy	snow
Cavity = true	0.144	0.02	0.016	0.02
Cavity = false	0.576	0.08	0.064	0.08

- Every question about a domain can be answered by the joint distribution

Conditional probability

- Conditional or posterior probabilities
  e.g., P(*cavity* | *toothache*) = 0.8
  i.e., given that *toothache* is all I know

- Notation for conditional distributions:
  P(*cavity* | *toothache*) = 2-element vector of 2-element vectors

- If we know more, e.g., *cavity* is also given, then we have
  P(*cavity* | *toothache, cavity*) = 1

- New evidence may be irrelevant, allowing simplification, e.g.,
  P(*cavity* | *toothache. sunny*) = P(*cavity* | *toothache*) = 0.8
  - This kind of inference, sanctioned by domain knowledge, is crucial

Conditional probability

- Definition of conditional probability:
  $P(a \mid b) = P(a \wedge b) / P(b)$ if $P(b) > 0$

- Product rule gives an alternative formulation:
  $P(a \wedge b) = P(a \mid b) P(b) = P(b \mid a) P(a)$

- A general version holds for whole distributions, e.g.,
  $\mathbf{P}(Weather, Cavity) = \mathbf{P}(Weather \mid Cavity)\, \mathbf{P}(Cavity)$
  - (View as a set of 4 × 2 equations, not matrix mult.)

- Chain rule is derived by successive application of product rule:

$$\begin{aligned}\mathbf{P}(X_1, \ldots, X_n) &= \mathbf{P}(X_1,\ldots,X_{n-1})\, \mathbf{P}(X_n \mid X_1,\ldots,X_{n-1}) \\ &= \mathbf{P}(X_1,\ldots,X_{n-2})\, \mathbf{P}(X_{n-1} \mid X_1,\ldots,X_{n-2})\, \mathbf{P}(X_n \mid X_1,\ldots,X_{n-1}) \\ &= \ldots \\ &= \pi_{i=1}^{n} \mathbf{P}(X_i \mid X_1, \ldots, X_{i-1})\end{aligned}$$

Inference by enumeration

- Start with the joint probability distribution:

	toothache		*¬ toothache*	
	catch	*¬ catch*	*catch*	*¬ catch*
cavity	.108	.012	.072	.008
¬ cavity	.016	.064	.144	.576

- For any proposition φ, sum the atomic events where it is true: $P(\varphi) = \Sigma_{\omega:\omega \models \varphi} P(\omega)$
- P(*toothache*) = ?

Inference by enumeration

- Start with the joint probability distribution:

	toothache		¬ *toothache*	
	catch	¬ *catch*	*catch*	¬ *catch*
cavity	.108	.012	.072	.008
¬ *cavity*	.016	.064	.144	.576

- For any proposition φ, sum the atomic events where it is true: $P(\varphi) = \Sigma_{\omega:\omega \models \varphi} P(\omega)$
- P(*toothache*) = 0.108 + 0.012 + 0.016 + 0.064 = 0.2
- P(*toothache* ∨ *cavity*) = ?

Inference by enumeration

- Start with the joint probability distribution:

	toothache		¬ *toothache*	
	catch	¬ *catch*	*catch*	¬ *catch*
cavity	.108	.012	.072	.008
¬ *cavity*	.016	.064	.144	.576

- For any proposition φ, sum the atomic events where it is true: $P(\varphi) = \Sigma_{\omega:\omega \models \varphi} P(\omega)$
- P(*toothache*) = 0.108 + 0.012 + 0.016 + 0.064 = 0.2
- P(*toothache* ∨ *cavity*) = 0.108 + 0.012 + 0.016 + 0.064 + 0.072 + 0.008 = 0.28
- P(¬ *cavity* | *toothache*) = ?

Inference by enumeration

- Start with the joint probability distribution:

	toothache		*¬ toothache*	
	catch	*¬ catch*	*catch*	*¬ catch*
cavity	.108	.012	.072	.008
¬ cavity	.016	.064	.144	.576

- Can also compute conditional probabilities:

$$P(\neg cavity \mid toothache) = \frac{P(\neg cavity \wedge toothache)}{P(toothache)}$$

$$= \frac{0.016+0.064}{0.108 + 0.012 + 0.016 + 0.064}$$

$$= 0.4$$

Normalization

	toothache		*¬ toothache*	
	catch	*¬ catch*	*catch*	*¬ catch*
cavity	.108	.012	.072	.008
¬ cavity	.016	.064	.144	.576

- Denominator can be viewed as a normalization constant α

P(*Cavity* | *toothache*) = α · **P**(*Cavity, toothache*)
= α · [**P**(*Cavity, toothache, catch*) + **P**(*Cavity, toothache, ¬catch*)]
= α · [<0.108, 0.016> + <0.012, 0.064>]
= α · <0.12, 0.08> = <0.6, 0.4>

General idea: compute distribution on query variable by fixing evidence variables and summing over hidden variables

Inference by enumeration, contd.

Typically, we are interested in
the posterior joint distribution of the query variables **Y**
given specific values **e** for the evidence variables **E**

Let the hidden variables be **H** = **X** - **Y** - **E**

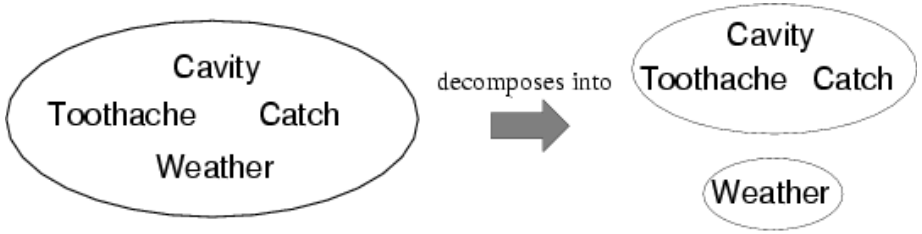
Then the required summation of joint entries is done by summing out the hidden variables:

$$\mathbf{P}(\mathbf{Y} \mid \mathbf{E} = \mathbf{e}) = \alpha \mathbf{P}(\mathbf{Y},\mathbf{E} = \mathbf{e}) = \alpha \Sigma_h \mathbf{P}(\mathbf{Y},\mathbf{E}= \mathbf{e}, \mathbf{H} = \mathbf{h})$$

- The terms in the summation are joint entries because **Y**, **E** and **H** together exhaust the set of random variables
- Obvious problems:
  1. Worst-case time complexity $O(d^n)$ where d is the largest arity
  2. Space complexity $O(d^n)$ to store the joint distribution
  3. How to find the numbers for $O(d^n)$ entries?

Independence

- *A* and *B* are independent iff
  P(*A*|*B*) = **P**(*A*)   or **P**(*B*|*A*) = **P**(*B*)   or **P**(A, B) = **P**(*A*) **P**(*B*)



P(*Toothache, Catch, Cavity, Weather*)
= **P**(*Toothache, Catch, Cavity*) **P**(*Weather*)

- 32 entries reduced to 12 (8+4);
  for *n* independent biased coins, $O(2^n) \rightarrow O(n)$
- Absolute independence powerful but rare
- Dentistry is a large field with hundreds of variables, none of which are independent. What to do?

Conditional independence

- $\mathbf{P}(Toothache, Cavity, Catch)$ has $2^3 - 1 = 7$ independent entries
- If I have a cavity, the probability that the probe catches in it doesn't depend on whether I have a toothache:
  (1) $\mathbf{P}(catch \mid toothache, cavity) = \mathbf{P}(catch \mid cavity)$
- The same independence holds if I haven't got a cavity:
  (2) $\mathbf{P}(catch \mid toothache, \neg cavity) = \mathbf{P}(catch \mid \neg cavity)$
- *Catch* is conditionally independent of *Toothache* given *Cavity*:
  $\mathbf{P}(Catch \mid Toothache, Cavity) = \mathbf{P}(Catch \mid Cavity)$
- Equivalent statements:
  $\mathbf{P}(Toothache \mid Catch, Cavity) = \mathbf{P}(Toothache \mid Cavity)$
  $\mathbf{P}(Toothache, Catch \mid Cavity) = \mathbf{P}(Toothache \mid Cavity)\ \mathbf{P}(Catch \mid Cavity)$

Conditional independence contd.

- Write out full joint distribution using chain rule:

  P(*Toothache, Catch, Cavity*)
  = **P**(*Toothache* | *Catch, Cavity*) **P**(*Catch, Cavity*)
  = **P**(*Toothache* | *Catch, Cavity*) **P**(*Catch* | *Cavity*) **P**(*Cavity*)
  = **P**(*Toothache* | *Cavity*) **P**(*Catch* | *Cavity*) **P**(Cavity)

  I.e., 2 + 2 + 1 = 5 independent numbers

- In most cases, the use of conditional independence reduces the size of the representation of the joint distribution from exponential in *n* to linear in *n*.

- Conditional independence is our most basic and robust form of knowledge about uncertain environments.

Bayes' Rule

- Product rule $P(a \wedge b) = P(a \mid b)\, P(b) = P(b \mid a)\, P(a)$
  $\Rightarrow$ Bayes' rule: $P(a \mid b) = P(b \mid a)\, P(a) / P(b)$

- or in distribution form

  $$\mathbf{P}(Y|X) = \mathbf{P}(X|Y)\, \mathbf{P}(Y) / \mathbf{P}(X) = \alpha\, \mathbf{P}(X|Y)\, \mathbf{P}(Y)$$

- Useful for assessing diagnostic probability from causal probability:

  - P(Cause|Effect) = P(Effect|Cause) P(Cause) / P(Effect)
  - E.g., let *M* be meningitis, *S* be stiff neck:
    $P(m|s) = P(s|m)\, P(m) / P(s) = 0.5 \times 0.0002 / 0.05 = 0.0002$
  - Note: posterior probability of meningitis still very small!

Bayes' Rule and conditional independence

P(*Cavity* | *toothache* ∧ *catch*)

= α · **P**(*toothache* ∧ *catch* | *Cavity*) **P**(*Cavity*)

= α · **P**(*toothache* | *Cavity*) **P**(*catch* | *Cavity*) **P**(*Cavity*)

- This is an example of a naïve Bayes model:

  $\mathbf{P}(\text{Cause}, \text{Effect}_1, \ldots, \text{Effect}_n) = \mathbf{P}(\text{Cause}) \prod_i \mathbf{P}(\text{Effect}_i | \text{Cause})$

- Total number of parameters is linear in *n*

Bayesian networks

Chapter 14

Sections 1 – 2

Bayesian networks

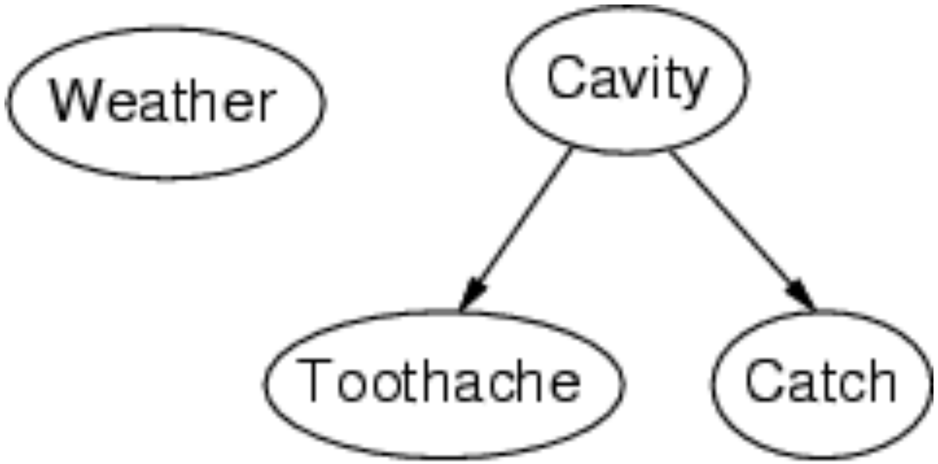
- A simple, graphical notation for conditional independence assertions and hence for compact specification of full joint distributions
- Syntax:
  - a set of nodes, one per variable
  - a directed, acyclic graph (link ≈ "directly influences")
  - a conditional distribution for each node given its parents:

$$\mathbf{P}(X_i \mid Parents(X_i))$$

- In the simplest case, conditional distribution represented as a conditional probability table (CPT) giving the distribution over X_i for each combination of parent values

Example

- Topology of network encodes conditional independence assertions:

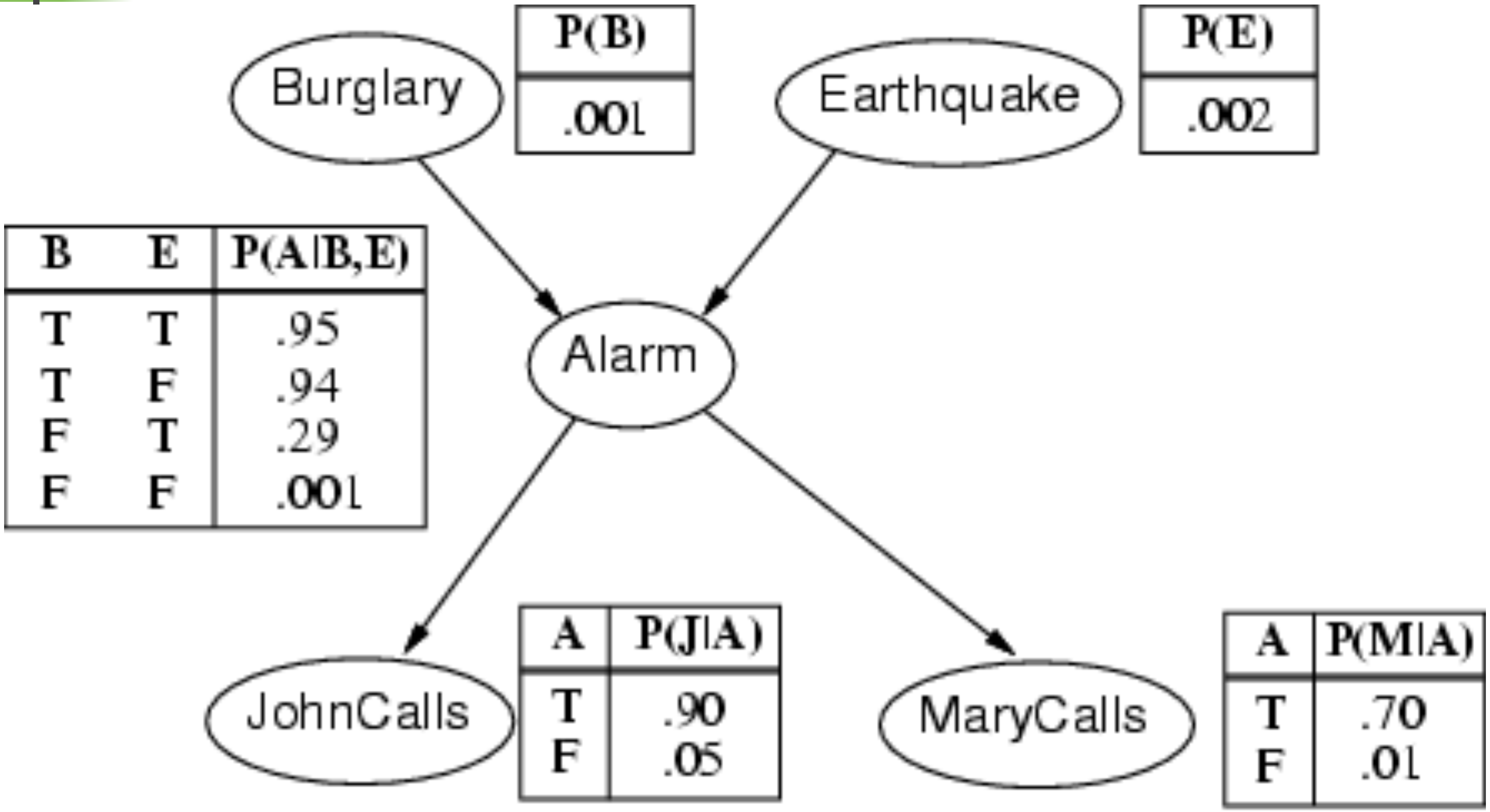


- *Weather* is independent of the other variables
- *Toothache* and *Catch* are conditionally independent given *Cavity*

Example

- I'm at work, neighbor John calls to say my alarm is ringing, but neighbor Mary doesn't call. Sometimes it's set off by minor earthquakes. Is there a burglar?
- Variables: *Burglary*, *Earthquake*, *Alarm*, *JohnCalls*, *MaryCalls*
- Network topology reflects "causal" knowledge:
  - A burglar can set the alarm off
  - An earthquake can set the alarm off
  - The alarm can cause Mary to call
  - The alarm can cause John to call

Example contd.



P(B)
.001

P(E)
.002

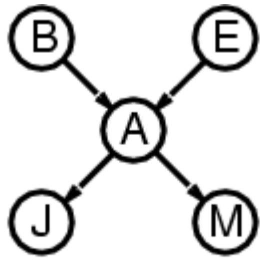
B	E	P(A|B,E)
T	T	.95
T	F	.94
F	T	.29
F	F	.001

A	P(J|A)
T	.90
F	.05

A	P(M|A)
T	.70
F	.01

Compactness

- A CPT for Boolean X_i with k Boolean parents has 2^k rows for the combinations of parent values

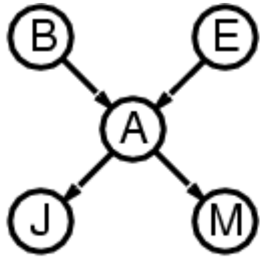


- Each row requires one number p for $X_i = true$
(the number for $X_i = false$ is just $1-p$)
- If each variable has no more than k parents, the complete network requires $O(n \cdot 2^k)$ numbers
- I.e., grows linearly with n, vs. $O(2^n)$ for the full joint distribution
- For burglary net, 1 + 1 + 4 + 2 + 2 = 10 numbers (vs. 2^5-1 = 31)

Semantics

The full joint distribution is defined as the product of the local conditional distributions:

$$\mathbf{P}(X_1, \ldots, X_n) = \prod_{i=1}^{n} \mathbf{P}(X_i \mid Parents(X_i))$$



e.g., $\mathbf{P}(j \wedge m \wedge a \wedge \neg b \wedge \neg e)$

$= \mathbf{P}(j \mid a)\, \mathbf{P}(m \mid a)\, \mathbf{P}(a \mid \neg b, \neg e)\, \mathbf{P}(\neg b)\, \mathbf{P}(\neg e)$

Constructing Bayesian networks

- 1. Choose an ordering of variables $X_1, \dots, X_n$
- 2. For $i = 1$ to n
  - add X_i to the network
  - select parents from $X_1, \dots, X_{i-1}$ such that

$$\mathbf{P}(X_i \mid Parents(X_i)) = \mathbf{P}(X_i \mid X_1, \dots X_{i-1})$$

This choice of parents guarantees

$$\mathbf{P}(X_1, \dots, X_n) = \pi_{i=1}^{n} \mathbf{P}(X_i \mid X_1, \dots, X_{i-1}) \quad \text{(chain rule)}$$

$$= \pi_{i=1}^{n} \mathbf{P}(X_i \mid Parents(X_i)) \quad \text{(by construction)}$$

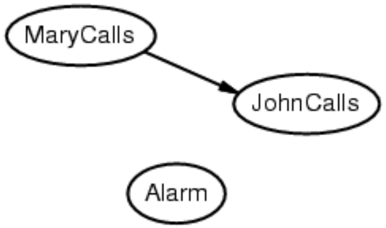
Example

- Suppose we choose the ordering *M, J, A, B, E*

P$(J \mid M) =$ ***P***(J)?

Example

- Suppose we choose the ordering *M, J, A, B, E*



$\mathbf{P}(J \mid M) = \mathbf{P}(J)$? **No**

$\mathbf{P}(A \mid J, M) = \mathbf{P}(A \mid J)$? $\mathbf{P}(A \mid J, M) = \mathbf{P}(A)$?

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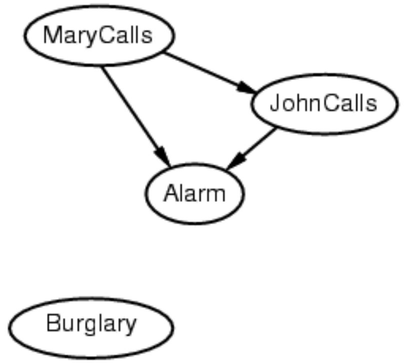

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$\mathbf{P}(B \mid A, J, M) = \mathbf{P}(B)$?

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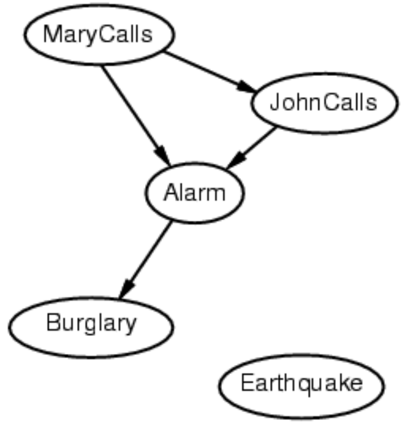

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$\mathbf{P}(A \mid J, M) = \mathbf{P}(A \mid J)$? $\mathbf{P}(A \mid J, M) = \mathbf{P}(A)$? **No**

$\mathbf{P}(B \mid A, J, M) = \mathbf{P}(B \mid A)$? **Yes**

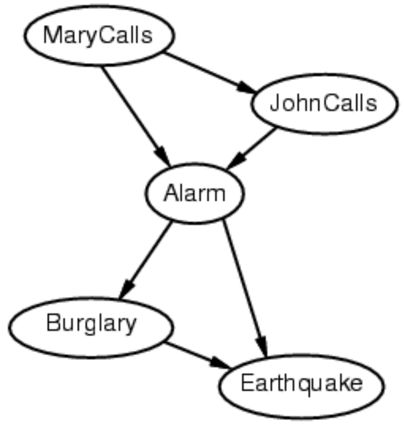
$\mathbf{P}(B \mid A, J, M) = \mathbf{P}(B)$? **No**

$\mathbf{P}(E \mid B, A ,J, M) = \mathbf{P}(E \mid A)$?

$\mathbf{P}(E \mid B, A, J, M) = \mathbf{P}(E \mid A, B)$?

Example

- Suppose we choose the ordering *M, J, A, B, E*



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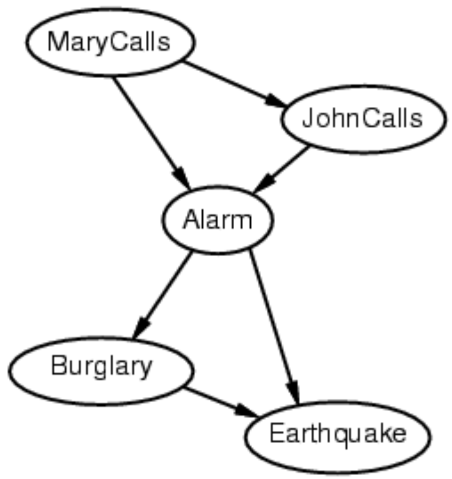
$\mathbf{P}(E \mid B, A, J, M) = \mathbf{P}(E \mid A, B)$? **Yes**

Example contd.



- Deciding conditional independence is hard in noncausal directions
- (Causal models and conditional independence seem hardwired for humans!)
- Network is less compact: 1 + 2 + 4 + 2 + 4 = 13 numbers needed

Summary

- Probability is a rigorous formalism for uncertain knowledge
- Joint probability distribution specifies probability of every atomic event
  - Queries can be answered by summing over atomic events
- For nontrivial domains, we must find a way to reduce the joint size
  - Independence and conditional independence provide the tools

Summary

- Bayesian networks provide a natural representation for (causally induced) conditional independence
- Topology + CPTs = compact representation of joint distribution