CS4234
Optimiz(s)ation Algorithms

L2 – Approximation Algorithms and Linear Programming

https://visualgo.net/en/mvc

(both unweighted and weighted version)

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For CS4234: probably the PS (high-level) solutions, tutorial, and past midterm/final **answers** that are preferably not stored in public domain that will be 'easily indexed by Google'

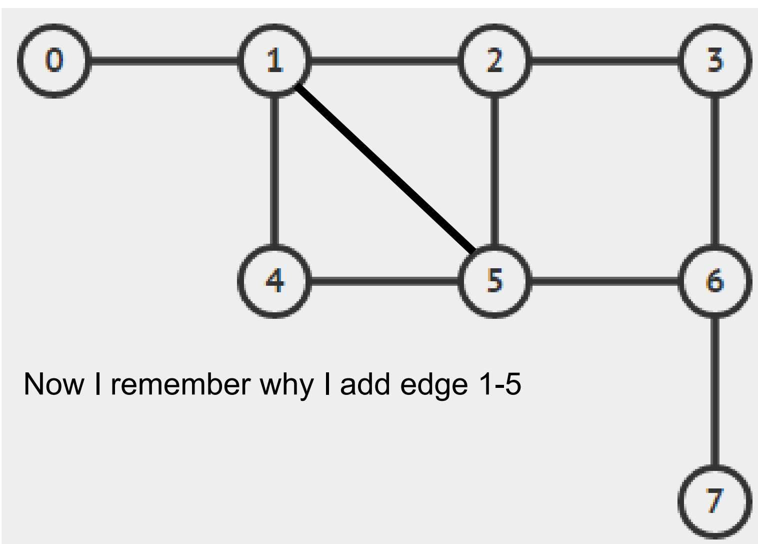
https://www.comp.nus.edu.sg/~stevenha/cs4234.html is public

Lecture 1 Recap (1)

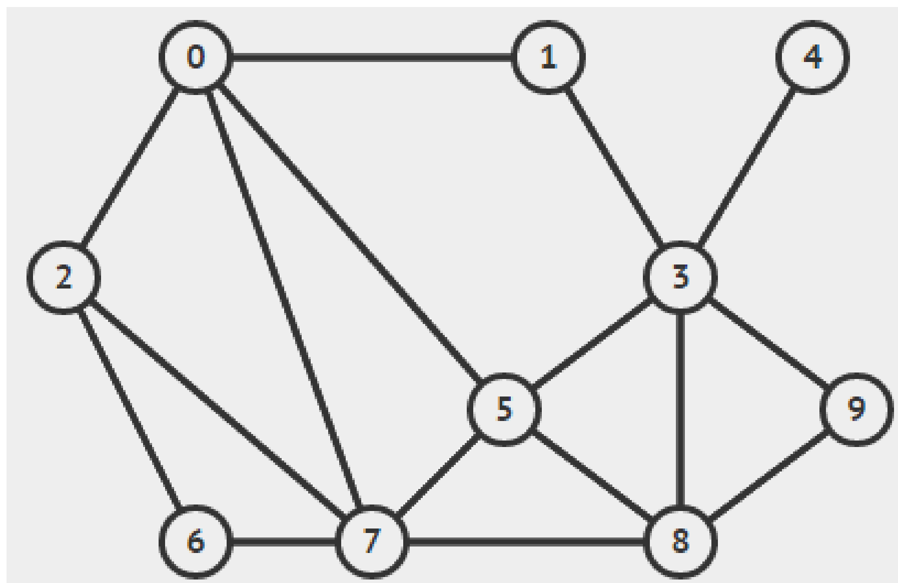
Definition of the first **Combinatorial Optimization Problem (COP)** in CS4234: `Min-Vertex-Cover`

- We revisit CS3230 to (re-)prove that VC is NP-Complete
- Thus MVC is NP-hard (as hard as VC) but not in NP

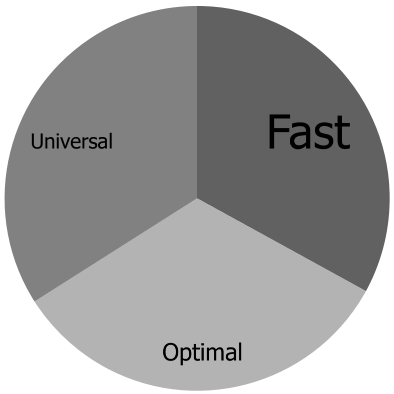
|MVC| = 4
example MVC = {1, 3, 5, 6}



|MVC| = 5 (Prof Halim will improve the bruteforce animation)
example MVC = {3, 0, 7, 8, 2}



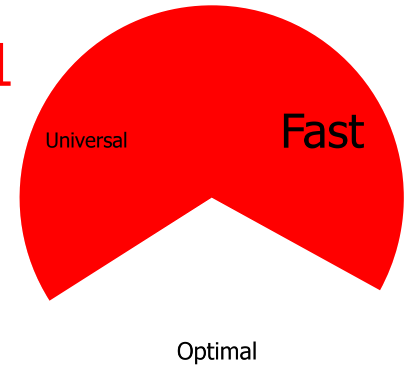
Lecture 1 Recap (2)



What should we do if we know (or we can reduce an known NP-complete problem into our current problem (the decision variant)

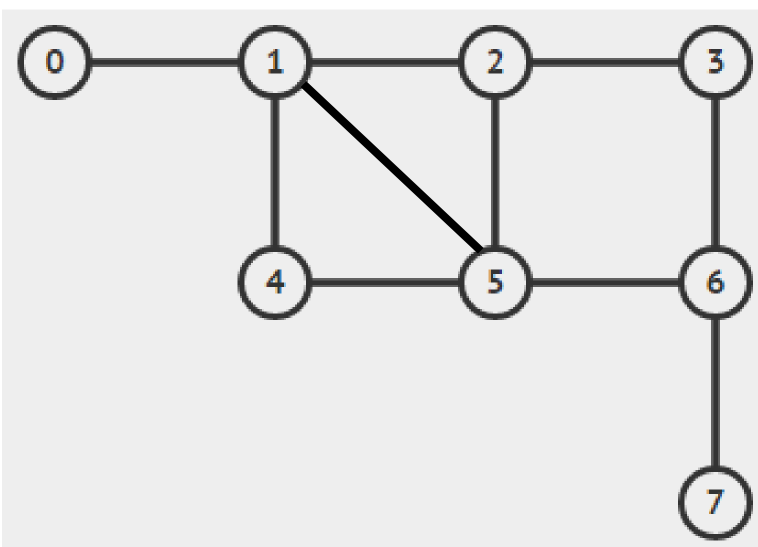
- Hope that it is posed on **special case** that somehow has polynomial solution(s) → lost **universality**
  - e.g., MVC on (Binary) Tree, we have DP solution (or? see T01)
- Hope that it is on **small** **instance/parameter** → still actually **not fast** (**exponential** time)
  - e.g., MVC on small k (or? see future questions)
- Hope that we can get by with **good enough** solution **fast** → good enough may be **non-optimal**
  - e.g., MVC, but we are OK if the answer is at most 2 x OPT
    - To be discussed now ☺

Approx Algo for MVC – Deterministic 1



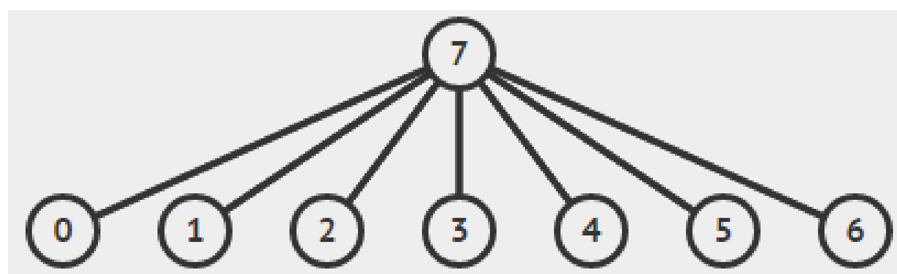

```
/* This algorithm adds vertices greedily, one at a time, until everything
   is covered.  The edges are considered in an arbitrary order, and for
   each edge, an arbitrary endpoint is added.                          */
Algorithm: ApproxVertexCover-1(G = (V, E))
Procedure:
    C ← ∅
    /* Repeat until every edge is covered:                             */
    while E ≠ ∅ do
        Let e = (u, v) be any edge in G.
        C ← C ∪ {u}
        G ← G_{-u} // Remove u and all adjacent edges from G.
    return C
```

Q: how to efficiently implement this?

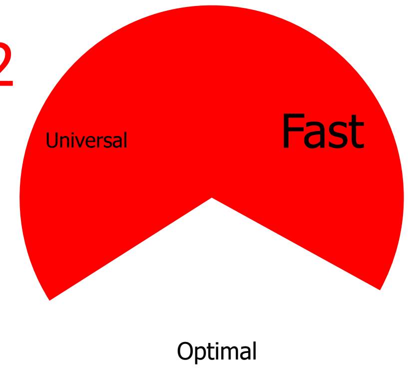


Can be very bad:

- **??**-approximation **(no bound)**



Approx Algo for MVC – Deterministic 2




```
/* This algorithm adds vertices greedily, two at a time, until everything
   is covered. The edges are considered in an arbitrary order, and for
   each edge, both endpoints are added.                                   */
```

Algorithm: ApproxVertexCover-2($G = (V, E)$)

Procedure:

$C \leftarrow \emptyset$

`/* Repeat until every edge is covered: */`

while $E \neq \emptyset$ **do**

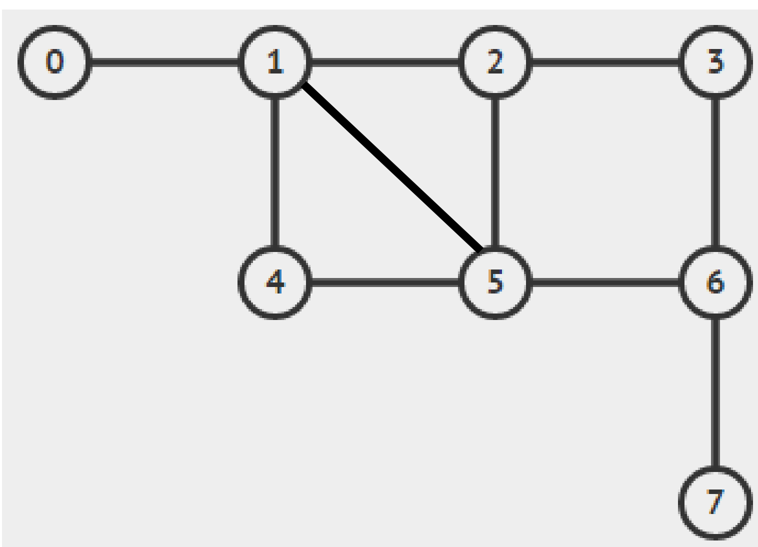
Let $e = (u, v)$ be any edge in G.

$C \leftarrow C \cup \{u, v\}$

$G \leftarrow G_{-\{u,v\}}$ `// Remove` u `and` v `and all adjacent edges from` G.

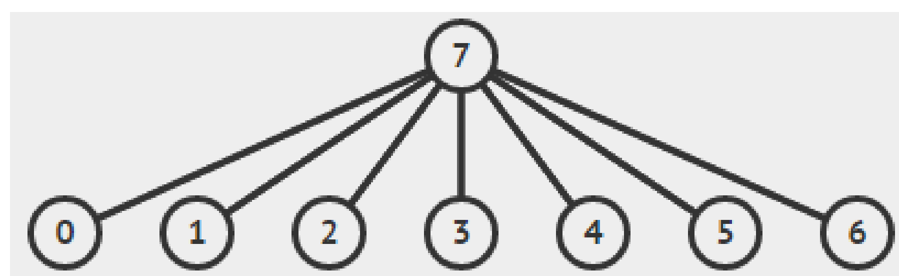
return C

Q: how to efficiently implement this?

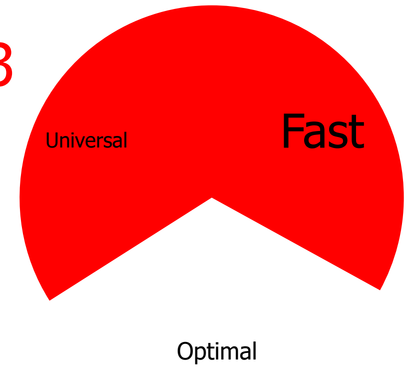


Not that bad?

– More analysis later



Approx Algo for MVC – Deterministic 3



/* This algorithm adds vertices greedily, one at a time, until everything is covered. At each step, the algorithm chooses the next vertex that will cover the most uncovered edges. */

Algorithm: ApproxVertexCover-3($G = (V, E)$)

Procedure:

$C \leftarrow \emptyset$

/* Repeat until every edge is covered: */

while $E \neq \emptyset$ **do**

Let $d(x)$ = number of uncovered edges adjacent to x.

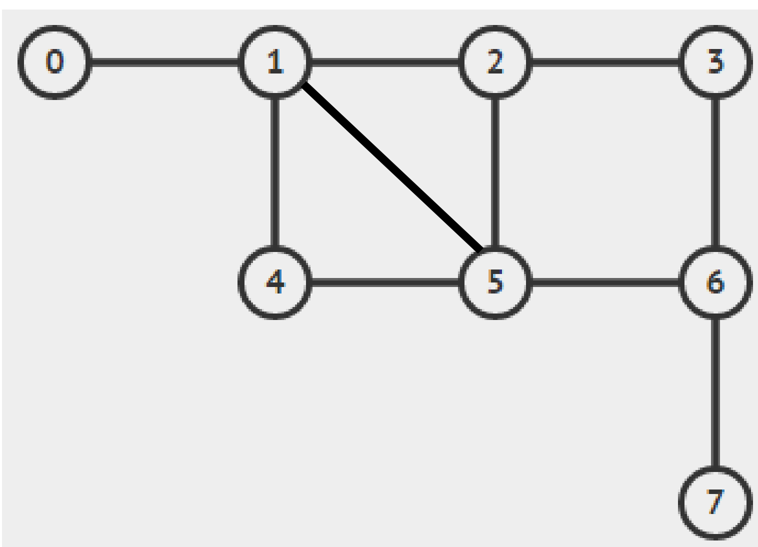
Let $u = \text{argmax}_{x \in V}\, d(x)$

$C \leftarrow C \cup \{u\}$

$G \leftarrow G_{-\{u\}}$ // Remove u and all adjacent edges from G.

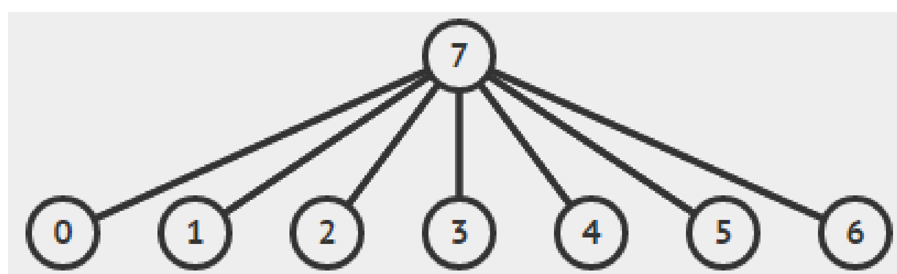
return C

Q: how to efficiently implement this?

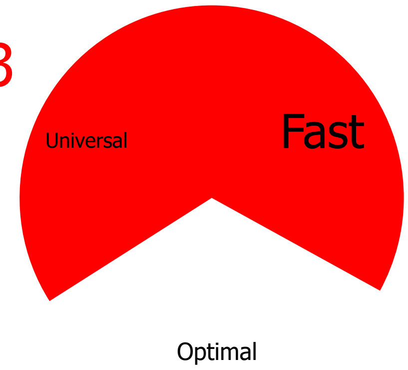


Not that bad?

– More analysis in the next slide



Approx Algo for MVC – Deterministic 3




```
/* This algorithm adds vertices greedily, one at a time, until everything
   is covered. At each step, the algorithm chooses the next vertex that
   will cover the most uncovered edges.                                  */
```

Algorithm: ApproxVertexCover-3($G = (V, E)$)

Procedure:

  $C \leftarrow \emptyset$

  `/* Repeat until every edge is covered: */`

  while $E \neq \emptyset$ **do**

    Let $d(x)$ = number of uncovered edges adjacent to x.

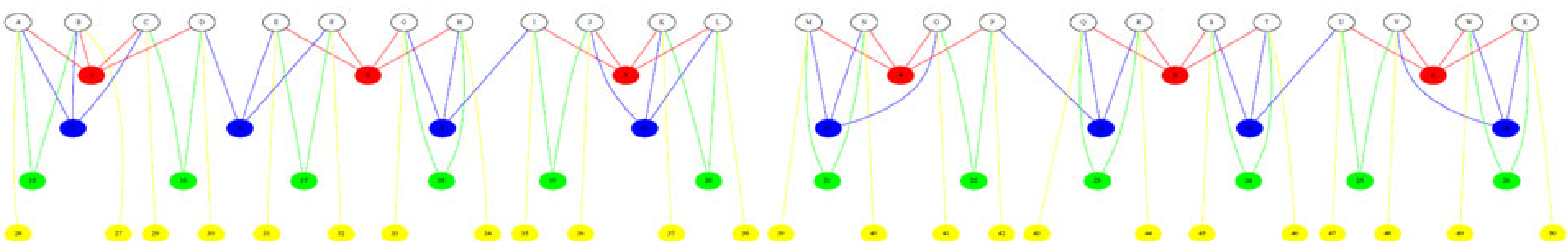
    Let $u = \mathrm{argmax}_{x \in V}\, d(x)$

    $C \leftarrow C \cup \{u\}$

    $G \leftarrow G_{-\{u\}}$ `// Remove` u `and all adjacent edges from` G.

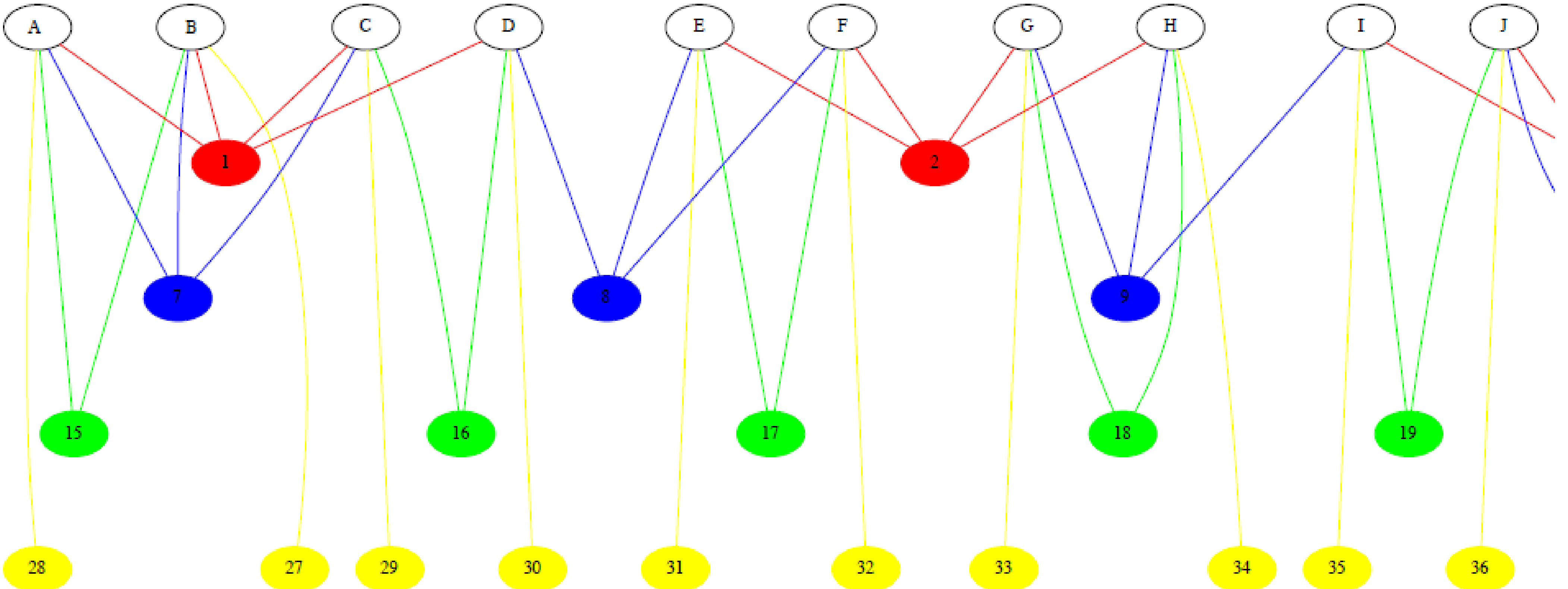
  return C

Below, **n** = 4, there are n! = 4! = 24 whites/yellows, 24/2 = 12 greens, 24/3 = 8 blues, and 24/4 = 6 reds



Zoomed-in

Optimal MVC = n! = 4! whites



The output of ApproxVertexCover-3 *could be*:

n!/n reds + n!/3 blues + n!/2 greens + n!/1 yellows =

n! * (1/n+1/3+1/2+1/1) = n! * (1/1+1/2+1/3+1/n) = n! * **ln n**

(Harmonic series)

This is ln-**n** (or log-**n**, base of log is negligible) factor worse than optimal answer

Approx Algo for MVC – Deterministic 2

/* This algorithm adds vertices greedily, two at a time, until everything is covered. The edges are considered in an arbitrary order, and for each edge, both endpoints are added. */

Algorithm: ApproxVertexCover-2$(G = (V, E))$

Procedure:

$C \leftarrow \emptyset$

/* Repeat until every edge is covered: */

while $E \neq \emptyset$ **do**

Let $e = (u, v)$ be any edge in G.

$C \leftarrow C \cup \{u, v\}$

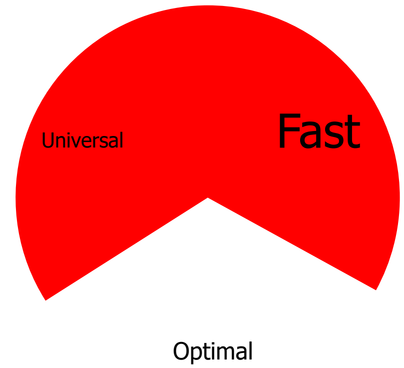
$G \leftarrow G_{-\{u,v\}}$ // Remove u and v and all adjacent edges from G.

return C

- This one has 2-approximation proof
  - Let **E'** be the edges considered by this algorithm, this **E'** is a **matching M** (revisited by Week 06 of CS4234, details in the PDF), **E' = M**
  - **|M|** ≤ **|OPT(G)|** (details in the PDF) and **C = 2*|E'|**
  - So **C = 2*|E'| = 2*|M|** ≤ **2*|OPT(G)|**

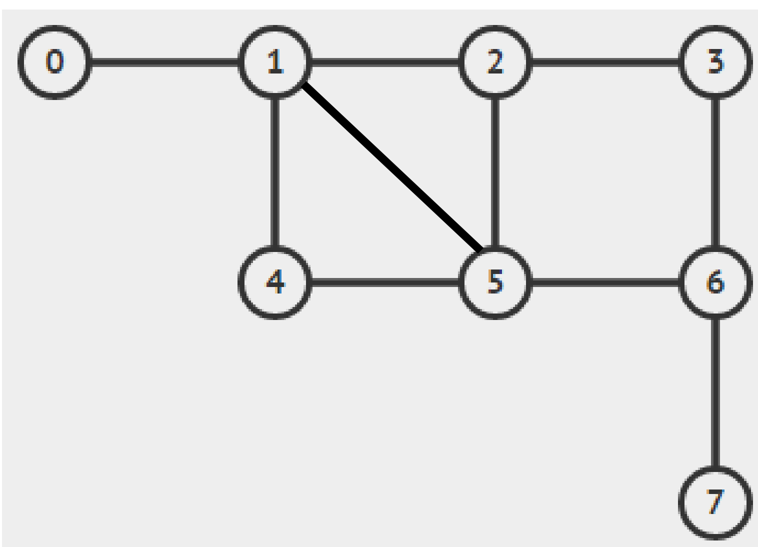
Try Deterministic 2-Opt live at https://visualgo.net/en/mvc

Approx Algo for MVC - Randomized




```
1  Algorithm: RandomizedVertexCover(G = (V, E))
2  Procedure:
3      C ← ∅
       /* Repeat until every edge is covered:                          */
4      while E ≠ ∅ do
5          Let e = (u, v) be any edge in G.
6          b ← Random(0, 1) // Returns {0, 1} each with probability 1/2.
7          if b = 0 then z = u
8          else if b = 1 then z = v
9          C ← C ∪ {z}
10         G ← G_{-z} // Remove z and all adjacent edges from G.
11     return C
```

Q: how to efficiently implement this?



2-approximation Analysis:

– Watered down proof:
- In line [5-8], the probability that we choose the right z is <u>at least ½</u>
- We include z in the cover and remove z (and 'cover' its edges) in line 9-10
- So C is at most 2 * OPT when E = ∅
  – In expectation *verbal discussion*

Try Probabilistic 2-Opt live at https://visualgo.net/en/mvc

In Practice…

- All approximation algorithms for MVC shown earlier run in polynomial time, i.e., **fast**
- No one is preventing you to run all of them and then report this ☺
  - [see the recording]

A high-level tour
Also in CP4 Book 2, p586-589

LINEAR PROGRAMMING

Linear Programming (LP)

A typical LP consists of three components:

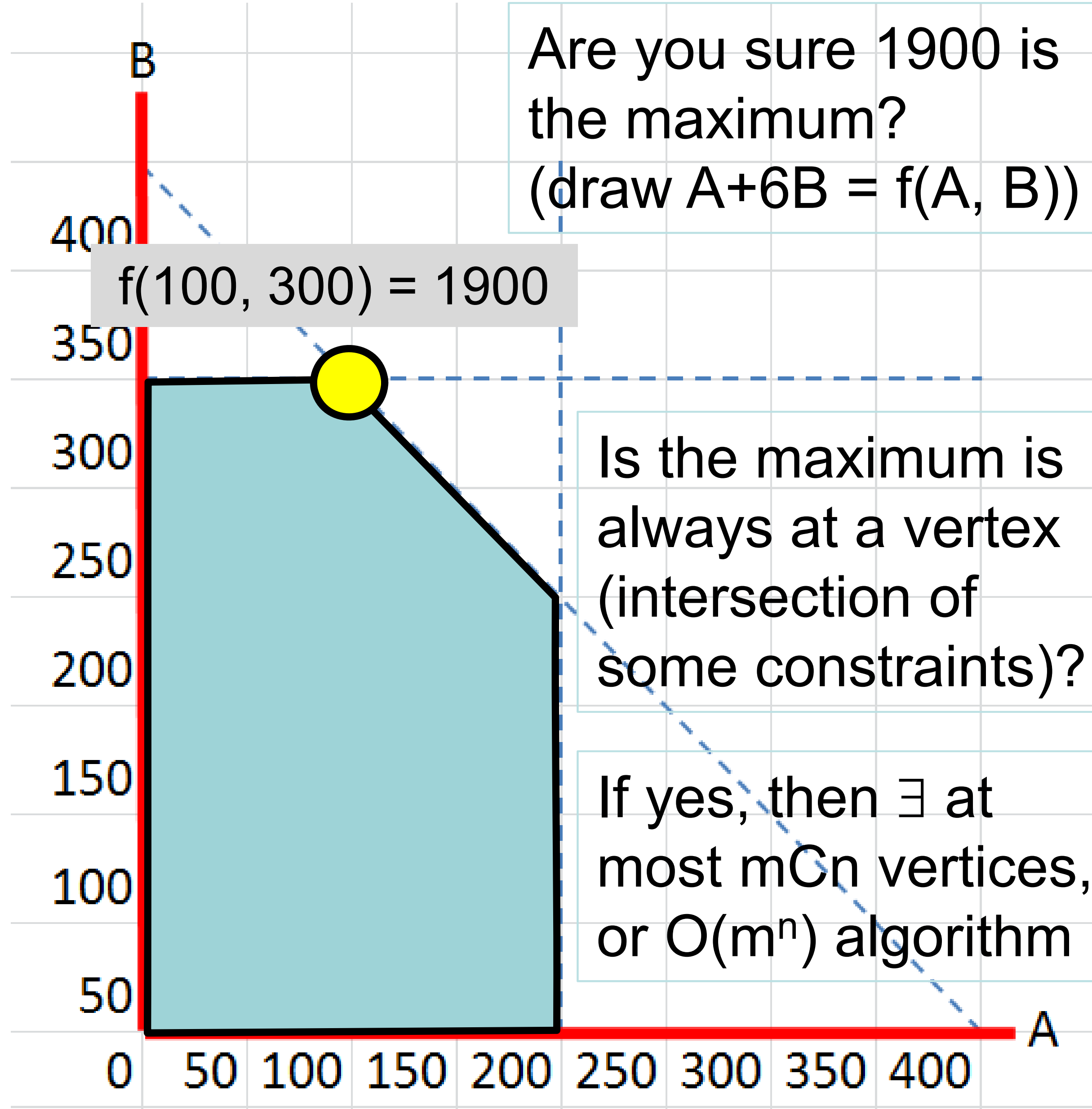
1. A list of (real-valued) variables $x_1, x_2, \ldots, x_n$
   - The goal: To find good values for these variables
2. An objective function $f(x_1, x_2, \ldots, x_n)$ that you are trying to maximize or minimize
   - The goal is to find the best values for the variables so as optimize this function
3. A set of m constraints that limits the feasible solution space
   - Each of these constraints is specified as an inequality

In a(n) LP problem, both the objective function and the constraints **are linear functions** of the variables

LP Example

A slightly different LP will be used in the "live" segment later

$$
\begin{aligned}
\max (A + 6B) \quad & \text{where:} \\
A &\leq 200 \\
B &\leq 300 \\
A + B &\leq 400 \\
A &\geq 0 \\
B &\geq 0
\end{aligned}
$$



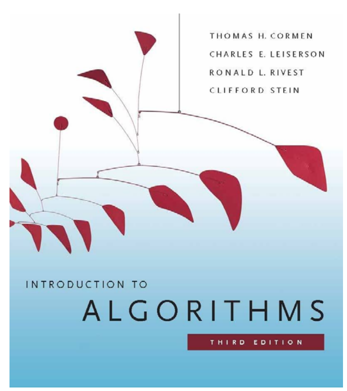
Are you sure 1900 is the maximum?
(draw A+6B = f(A, B))

Is the maximum is always at a vertex (intersection of some constraints)?

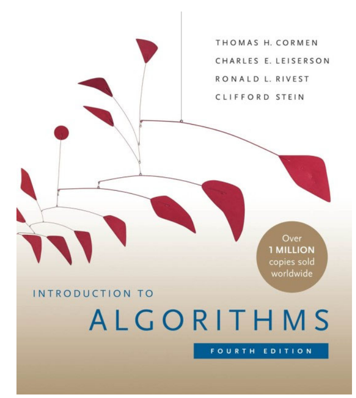
If yes, then ∃ at most mCn vertices, or $O(m^n)$ algorithm

Simplex Method

1. Find any (feasible) vertex v
2. Examine all the neighboring vertices of v: $v_1, v_2, \ldots, v_k$
3. Calculate $f(v), f(v_1), f(v_2), \ldots, f(v_k)$
   – If f(v) is the maximum (among its neighbors), then stop and return v
4. Otherwise, choose **<u>one of</u>** the neighboring vertices v_j where $f(v_j) > f(v)$
   – Let $v = v_j$
5. Go to step (2)

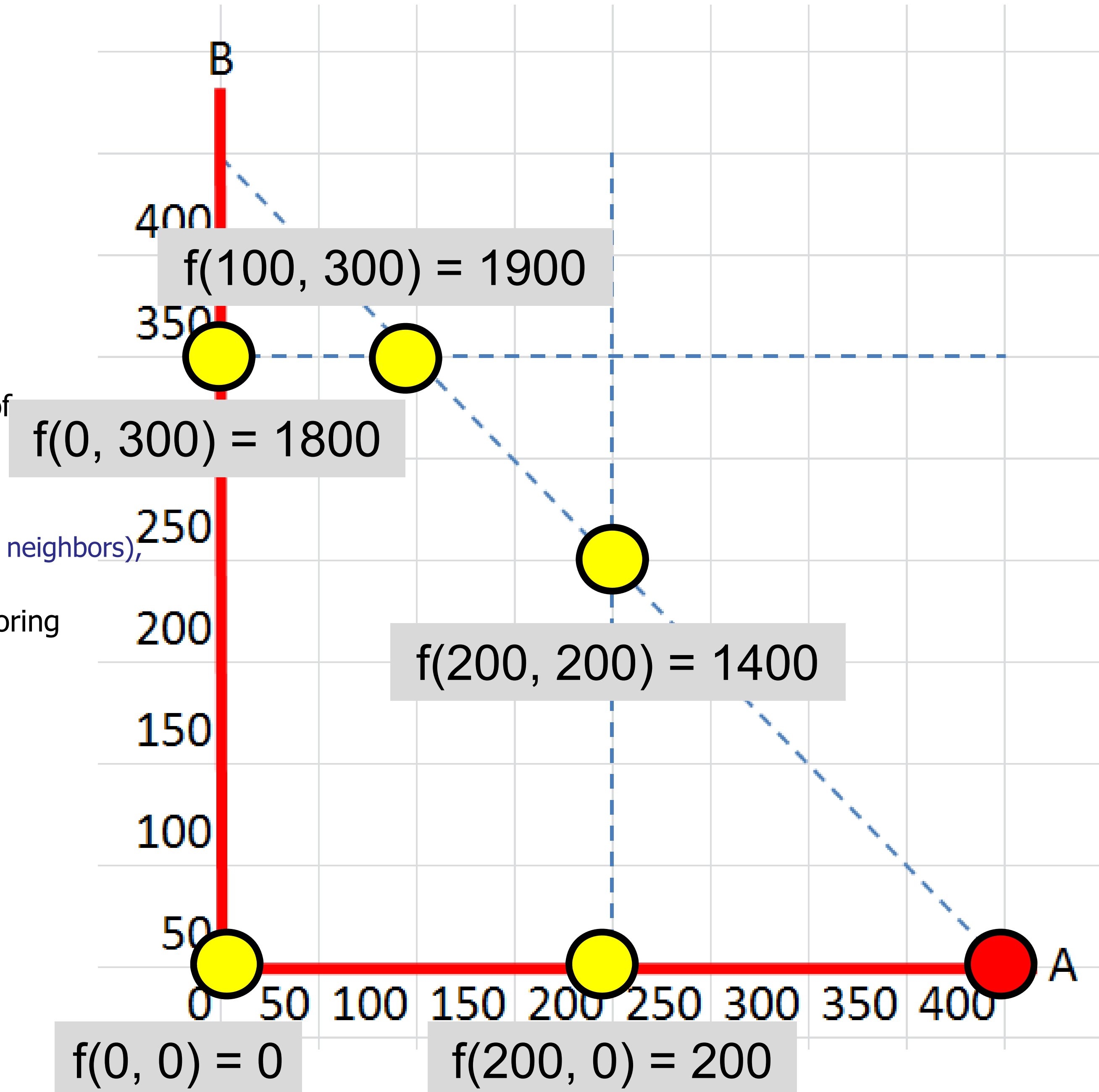


Details of Simplex is “dropped” in CLRS 4th edition…



Simplex Live Example

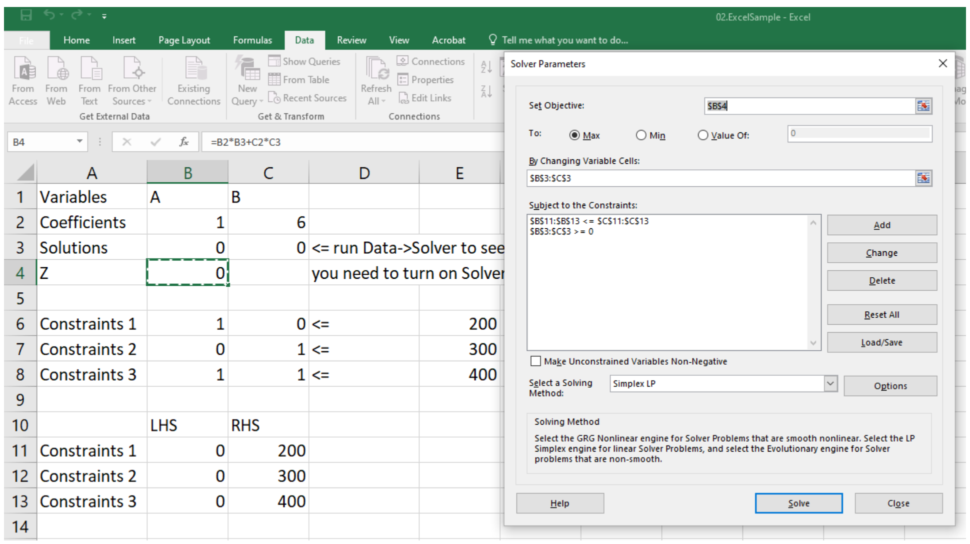
1. Find any (feasible) vertex v
2. Examine all the neighboring vertices of v $v_1, v_2, \ldots, v_k$
3. Calculate $f(v), f(v_1), f(v_2), \ldots, f(v_k)$
   - If f(v) is the maximum (among its neighbors), then stop and return v
4. Otherwise, choose **one of** the neighboring vertices v_j where $f(v_j) > f(v)$
   - Let $v = v_j$
5. Go to step (2)

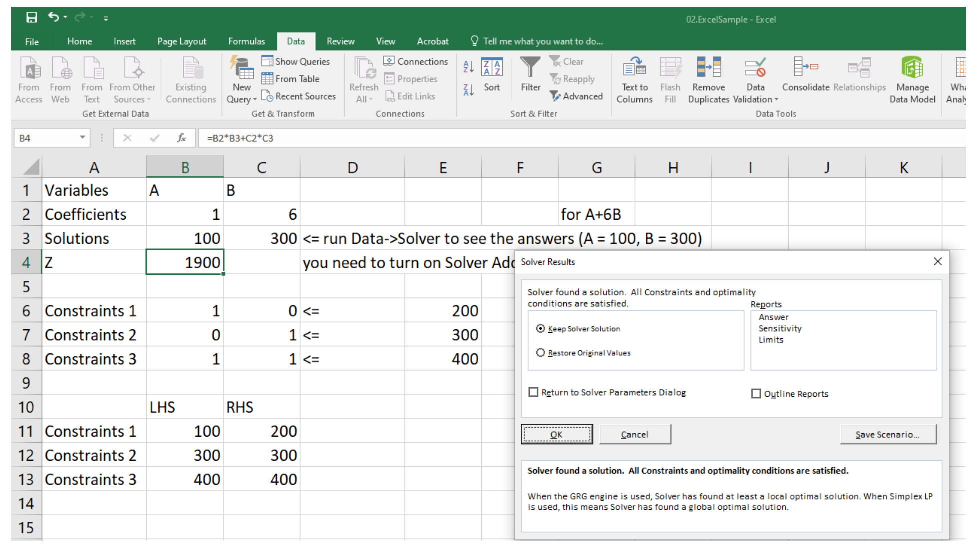


LP Solver in Microsoft Excel

Live Demonstration
(MS Excel can be setup quickly to use Solver Add-in)

See **02.ExcelSample.xlsx** (tab 'LP')





lp_solve in Ubuntu

Cannot do live demonstration on Steven's laptop…
it is on Windows ☹

If I cannot SSH to my DO droplet, just see this screenshot

```
[redacted]l-droplet:/# cat cs4234.lp
/* from CS4234 lecture note 02 */

max: A + 6 B;

A <= 200;
B <= 300;
A + B <= 400;
A >= 0;
B >= 0;
[redacted]l-droplet:/# lp_solve cs4234.lp

Value of objective function: 1900.00000000

Actual values of the variables:
A                              100
B                              300
```

Usable Simplex in C++

https://github.com/jaehyunp/stanfordacm/blob/master/code/Simplex.cc

Good usable Simplex code from Stanford ICPC team ☺

I have a local copy of that Simplex code in Java
(thanks to a senior student from "many" AYs ago)

Any volunteer to convert this to Python?

- Probably ChatGPT can do the translation too
- for non-Kattis projects, you may want to use https://docs.scipy.org/doc/scipy/reference/optimize.linprog-simplex.html

Now use any tool to solve this

- Maximize $777x+7y$
- Such that:
  - $100x \leq 15000$
  - $50y \leq 10000$
  - $x+y \leq 300$
- And x and y are non-negative

Answer for this year
[do it yourself!!]

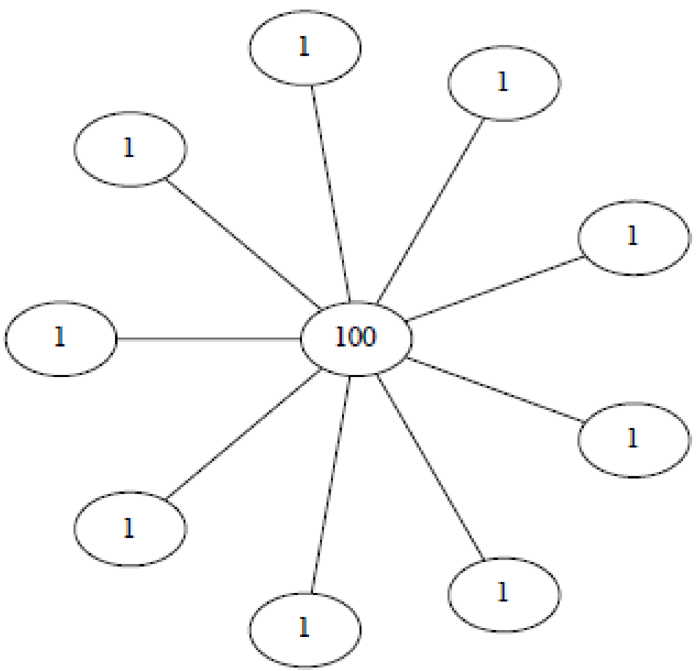
PS: Steven shall randomize the LP exercise each year

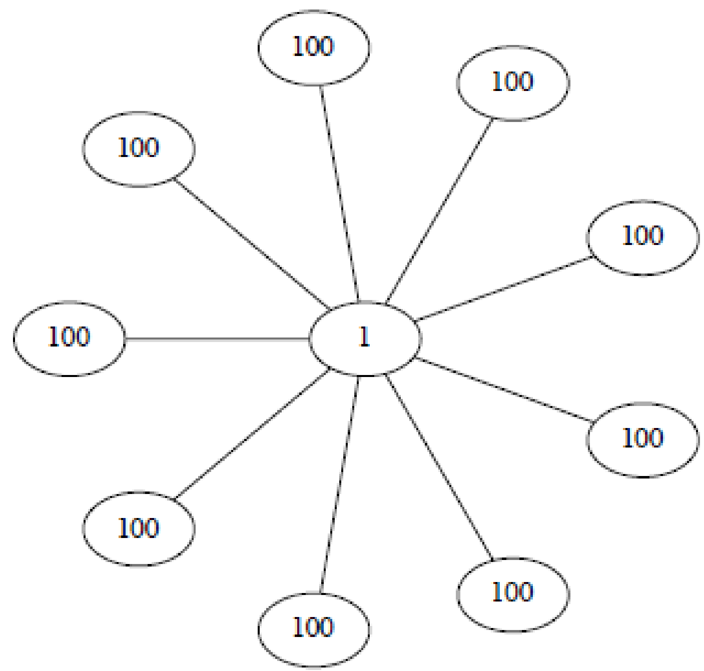
Two ways to ***approximately*** deal with this version…

MIN WEIGHT VERTEX COVER (MWVC)

MIN-WEIGHT-VERTEX-COVER

Both the Deterministic & Randomized 2-approximation algorithm for `Min-Vertex-Cover` "fail" on the weighted version; Do you understand why?





MWVC as an (Integer) Linear Program ILP/IP

The formulation (x_j is a Boolean {0, 1} variable where 0 = not in VC and 1 = in VC):

$$\min \left(\sum_{j=1}^{n} w(v_j) \cdot x_j \right) \quad \text{where:}$$

$$\begin{aligned} x_i + x_j &\geq 1 \quad \text{for all } (i,j) \in E \\ x_j &\geq 0 \quad \text{for all } j \in V \\ x_j &\leq 1 \quad \text{for all } j \in V \\ x_j &\in \mathbb{Z} \quad \text{for all } j \in V \end{aligned}$$

Notice that this line actually has E copies in an actual (I)LP program

Some other textbook says $x_j \in \{0, 1\}$, and there are V copies

The unweighted one
Proven NP-hard last week

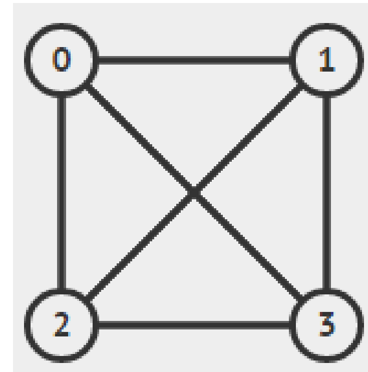
Min-Vertex-Cover (set $w(v_j)$ to all 1) $\leq_p$ ILP

So ILP is also NP-hard

PS: A tool for ILP that I have explored in the past

MWVC as a Relaxed Linear Program

Relaxing the Integer constraint



Assume w is all 1
Example LP solution
$x_0 = x_1 = x_2 = x_3 = 0.5$
What should we do?

$$\min\left(\sum_{j=1}^{n} w(v_j)\cdot x_j\right) \qquad \text{where:}$$

$$\begin{array}{rcll} x_i + x_j & \geq & 1 & \text{for all } (i,j) \in E \\ x_j & \geq & 0 & \text{for all } j \in V \\ x_j & \leq & 1 & \text{for all } j \in V \end{array}$$

[redacted]

PS: LP solution can be equal or better than ILP solution. Why?

Round up $\mathbf{x_j}$ value if it is ≥ 0.5

let $y_j = 1$ if $x_j \geq 1/2$, and let $y_j = 0$ otherwise.

But is this a good approximation?

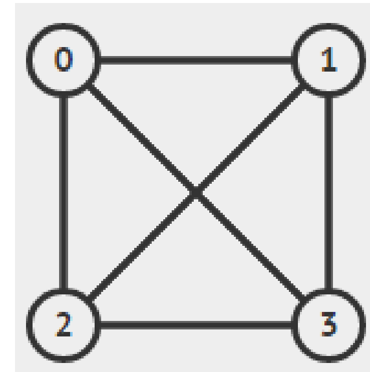
lp_solve output

```
[redacted] # cat k4.lp
min: x0+x1+x2+x3;

x0+x1 >= 1;
x0+x2 >= 1;
x0+x3 >= 1;
x1+x2 >= 1;
x1+x3 >= 1;
x2+x3 >= 1;

x0 >= 0;
x1 >= 0;
x2 >= 0;
x3 >= 0;

x0 <= 1;
x1 <= 1;
x2 <= 1;
x3 <= 1;
[redacted] lp_solve k4.lp

Value of objective function: 2.00000000

Actual values of the variables:
x0                              0.5
x1                              0.5
x2                              0.5
x3                              0.5
```

Analysis

OPT(G) = OPT(ILP) ≥ OPT(LP)

$$cost(OPT) \geq \sum_{j=1}^{n} w(v_j) \cdot x_j$$



Assume w is all 1

Example ILP solution $x_0 = x_1 = x_2 = 1;\ x_3 = 0$

Example LP solution $x_0 = x_1 = x_2 = x_3 = 0.5$

Rounded answer ≤ 2 x OPT(LP)

$$\begin{aligned} \sum_{j=1}^{n} w(v_j) \cdot \boxed{y_j} &\leq \sum_{j=1}^{n} w(v_j) \cdot \boxed{(2x_j)} \\ &\leq 2\left(\sum_{j=1}^{n} w(v_j) \cdot x_j\right) \\ &\leq 2 \times OPT(G) \end{aligned}$$

Notice, however, that $y_j \leq 2x_j$, for all j.

Analysis: This is a **2**-Approximation algorithm

Linear Programming Summary

General form of an LP:

1. A set of variables: $x_1, x_2, \ldots, x_n$
2. A linear objective to maximize (or minimize): $c^T x$
   - c and x as vectors,
   - c^T represents the transpose of c
   - multiplication represents the dot product
3. A set of linear constraints written as a Matrix equation: $Ax \leq b$

Presented as: $\mathbf{max}\ \mathbf{c^T x}$ **where** $\mathbf{Ax \leq b}$ **and** $\mathbf{x \geq 0}$

LP in Standard Form

Exercise to translate given LP into standard form

$$\begin{aligned} \min x_1 + 2x_2 - x_3 & \quad & \text{where // this is a minimization problem} \\ x_1 + x_2 & = & 7 \text{ // this is an equality} \\ x_2 - 2x_3 & \geq & 4 \text{ // this is a } \geq \text{ inequality} \\ x_1 & \leq & 2 \end{aligned}$$

Details in the PDF

Wait…

- There is an even simpler 2-Approximation algorithm for MWVC… which is…
  - See https://visualgo.net/en/mvc, M weighted VC

Summary

- Approximation algorithms for MVC
  - Deterministic, 3 variants, **but only variant 2 is 2-Approximation**
  - Randomized, ***Expected*** 2-Approximation
- Introduction to LP and an overview of Simplex Method
  - Simplex in Excel++, Ubuntu lp_solve, and custom code
- Introducing the weighted MVC (MWVC)
  - Problem with MVC approximation algorithms…
  - Reducing MWVC to ILP
  - Relaxing ILP to LP (we can use Simplex) and rounding up the answer
  - Analysis of that solution: **2-approximation**
  - Plus yet another alternative 2-approximation solution

Admin

- PS1 is open until this **Sun, 27 Aug 23, 11.59pm**
  - As of Thu, 24 August 2023, 11.30am…
    - 47 with ACs on A+B+C+D or more,
      - Ignore E+F+G+H if you have (many) other things to do
    - 7 more with only ≥ 2/4 ACs
      - Should also be on track to complete PS1, but will have a busy day today/tomorrow…
    - **But a staggering 69-47-7 = 15 pax with only 1 or 0 AC…**
      - Are you staying in this course or not?
  - Consider that first PS1, these first two lectures (,plus my Lecture 03a+03b recordings and future Tut01+Tut02) and decide if the (optional/elective) CS4234 is for you…
    - I am OK with if you drop* (Terms and conditions apply)
- PS2 still starts from **Sat, 26 Aug 23, 08.00am**
  - Four (:O) **NP-hard** optimization problems
- Tut01.pdf is out; first tutorial next **Mon, 28 Aug 23**

PS1 (More) Hints

- Only shown in the recording