CS4234
Optimiz(s)ation Algorithms

L3a – Min-Set-Cover

No VisuAlgo page yet

Any taker?

(1 current FYP student AY23/24 is probably going to be doing this soon)

MIN-SET-COVER (one more COP)

Combinatorial Optimization Problem

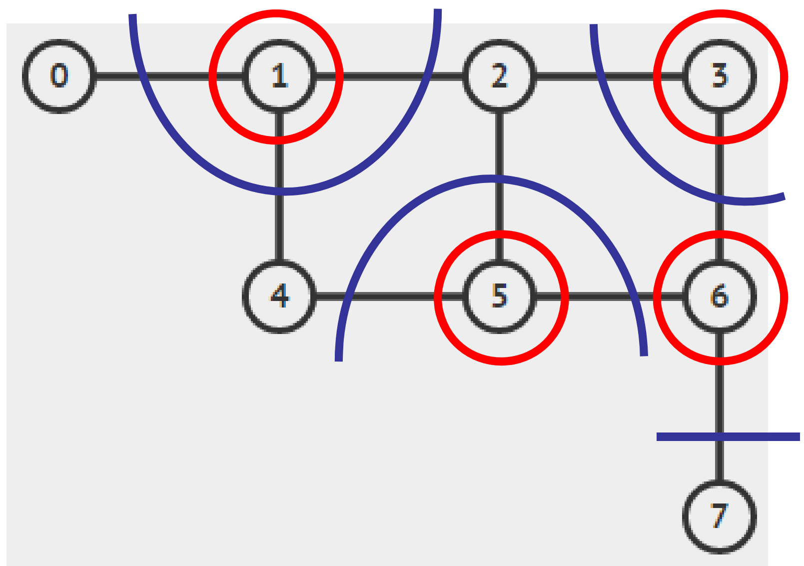
- Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of **n** elements
- Let $S_1, S_2, \ldots, S_{\mathbf{m}}$ be subsets of X, i.e., each $S_j \subseteq X$
  - Assume that every item in X appears in some set, i.e., $\cup_j S_j = X$
- A **set cover** of X with S is a set $I \subseteq \{1, 2, \ldots, \mathbf{m}\}$ such that $\cup_{j \in I} S_j = X$

Notice… $\exists\ 2^m$ possible such subsets

- The solution for **MIN-SET-COVER** problem is a **set cover I of minimum size**

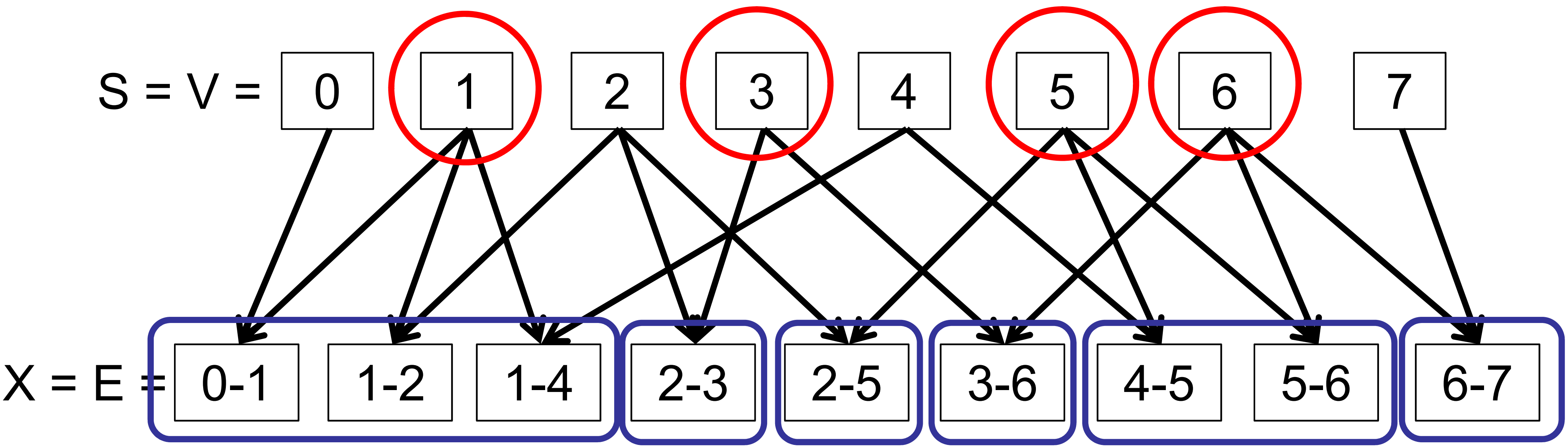
MIN-SET-COVER Example 1

PS:
I will add edge 1-5 from Lecture 1…
As this picture is actually a *bipartite*, not a *general* graph
I will make it consistent (one day)



VC $\leq_p$ SC…, so both are NP-hard

|VC| = |SC| = 4 in this example

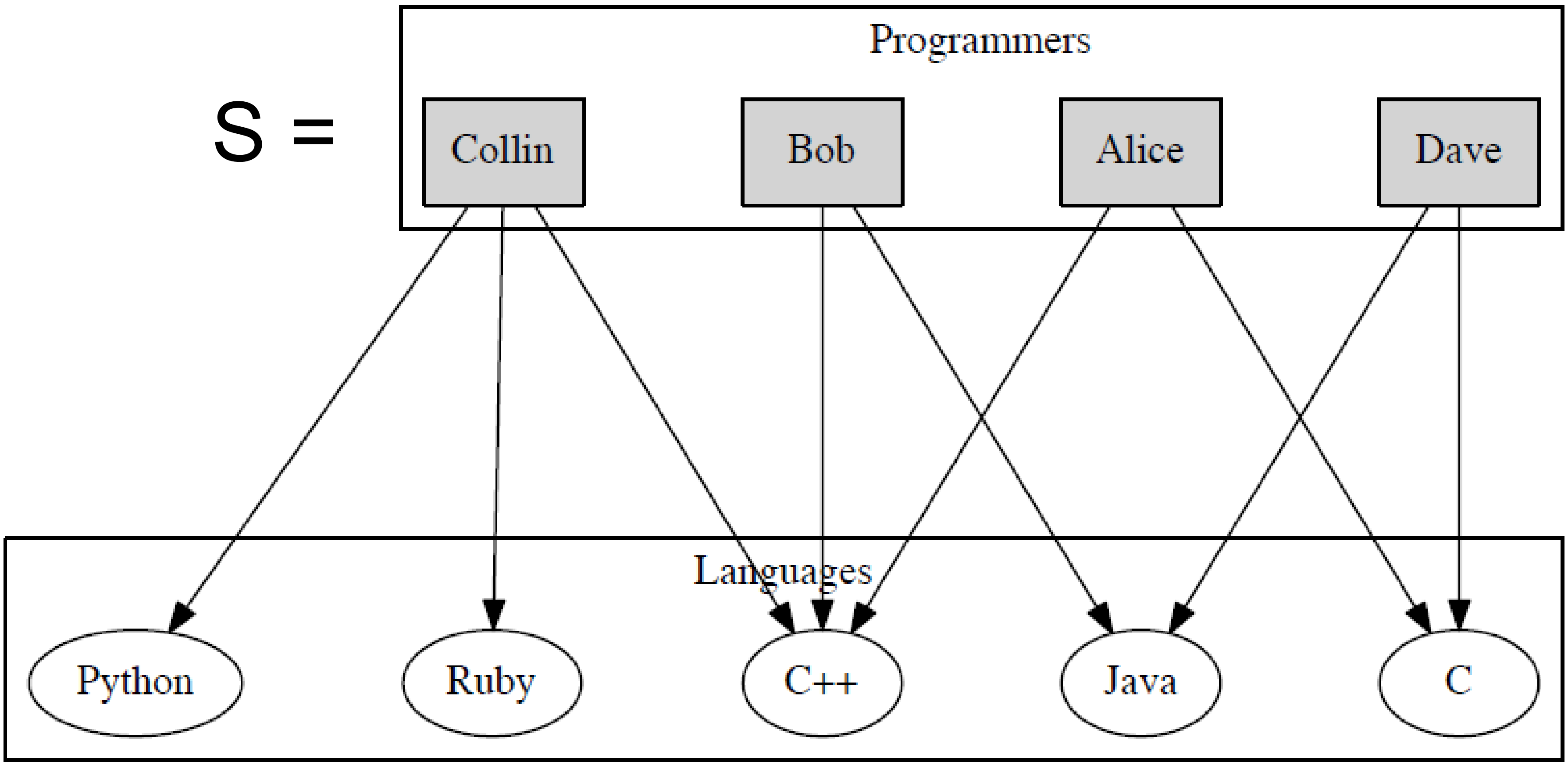


MIN-SET-COVER Example 2

Cost Saving in Software Company
What is the optimal solution?

S =

X =



MIN-SET-COVER Example 3

Stating Steven's problem for LAST AY's CS4234 tutorial group issues into an MSC problem? (not an issue this sem)

$n = |X| = 31$

3 groups

Group	Students	Timetable
T1 (CS4234)	12	Session 1 - Monday, Time: 11:00 - 12:00, Venue: SR_LT19, Recurrence: 13 (Steven Halim)
T2 (CS4234)	9	Session 1 - Monday, Time: 14:00 - 15:00, Venue: SR_LT19, Recurrence: 13 (Steven Halim)
T3 (CS4234)	10	Session 1 - Monday, Time: 17:00 - 18:00, Venue: SR_LT19, Recurrence: 13 (Steven Halim)

Review the recording for NUS students ☺

Now a “Doable” Task?

- https://nus.kattis.com/problems/socialadvertising

Review the recording for NUS students ☺

NOT-live coding (C++)

~~Steven will attempt to live code from here~~

~~Without compiling :O..., as he is not sure what devtools are available in LT19 desktop PC~~

~~Hopefully AC~~

Ah this is just a recording this year...
So I will show you my C++ code directly

GreedySetCover – A Greedy Algorithm

/* This algorithm adds sets greedily, one at a time, until everything is covered. At each step, the algorithm chooses the next set that will cover the most uncovered elements. */

Algorithm: GreedySetCover($X, S_1, S_2, \ldots, S_m$)

Procedure:

  $I \leftarrow \emptyset$

  /* Repeat until every element in X is covered: */

  while $X \neq \emptyset$ **do**

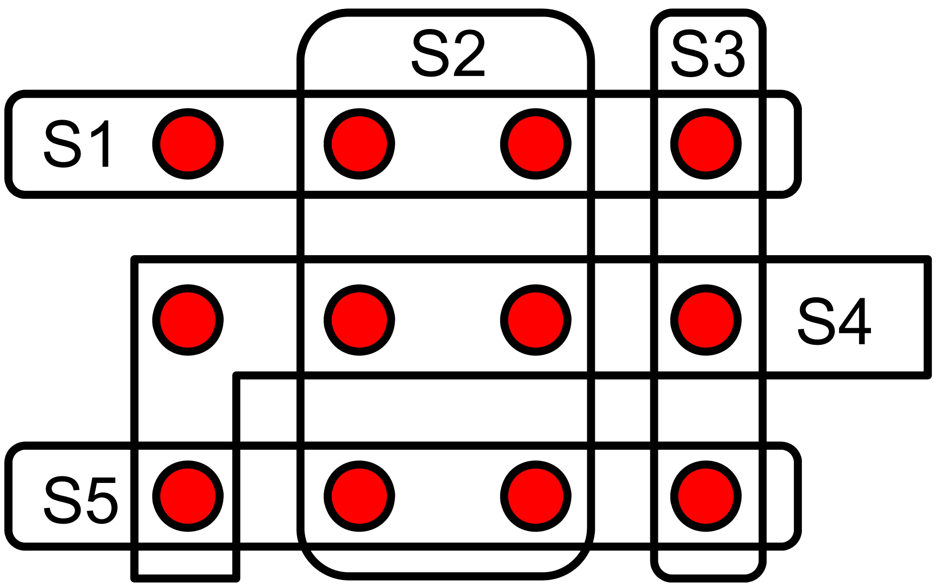
    Let $d(j) = |S_j \cap X|$ // This is the number of uncovered elements in S_j

    Let $j = \mathrm{argmax}_{i \in \{1,2,\ldots,m\}} d(i)$ // Break ties by taking lower i

    $I \leftarrow I \cup \{j\}$ // Include set S_j into the set cover

    $X \leftarrow X \setminus S_j$ // Remove elements in S_j from X.

  return I



GreedySetCover Execution (1)

/* This algorithm adds sets greedily, one at a time, until everything is covered. At each step, the algorithm chooses the next set that will cover the most uncovered elements. */

Algorithm: GreedySetCover($X, S_1, S_2, \ldots, S_m$)

Procedure:

  $I \leftarrow \emptyset$

  /* Repeat until every element in X is covered: */

  while $X \neq \emptyset$ **do**

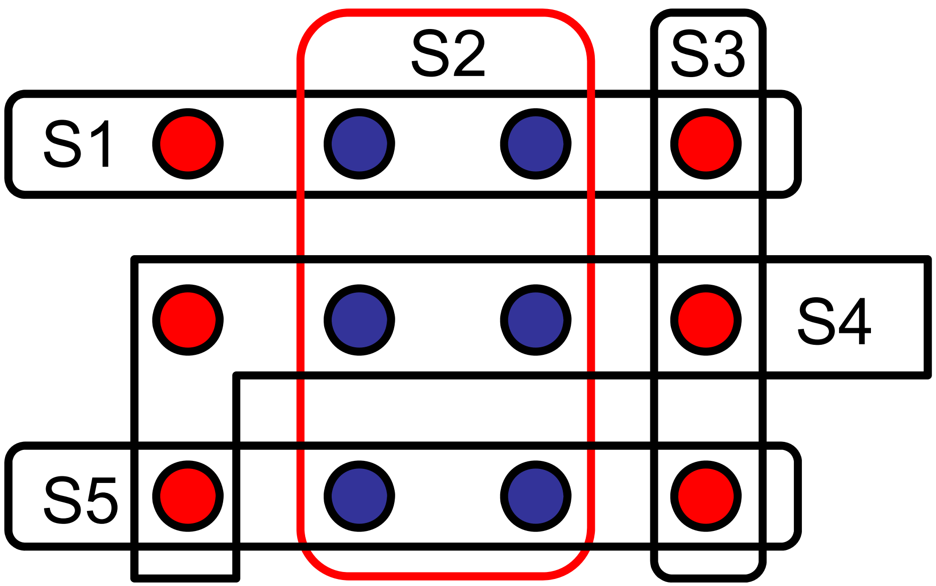
    Let $d(j) = |S_j \cap X|$ // This is the number of uncovered elements in S_j

    Let $j = \text{argmax}_{i \in \{1,2,\ldots,m\}} d(i)$ // Break ties by taking lower i

    $I \leftarrow I \cup \{j\}$ // Include set S_j into the set cover

    $X \leftarrow X \setminus S_j$ // Remove elements in S_j from X.

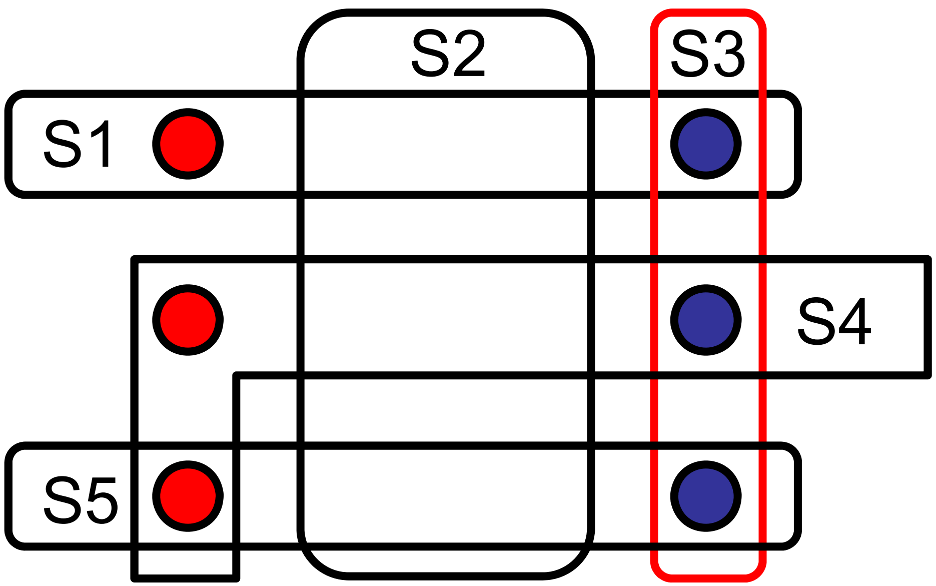
  return I



Set	d(j) - 1				
S1	4				
S2	6				
S3	3				
S4	5				
S5	4				

GreedySetCover Execution (2)

```
/* This algorithm adds sets greedily, one at a time, until everything is
   covered. At each step, the algorithm chooses the next set that will
   cover the most uncovered elements.                                  */
Algorithm: GreedySetCover(X, S_1, S_2, ..., S_m)
Procedure:
    I ← ∅
    /* Repeat until every element in X is covered:                     */
    while X ≠ ∅ do
        Let d(j) = |S_j ∩ X| // This is the number of uncovered elements in S_j
        Let j = argmax_{i∈{1,2,...,m}} d(i) // Break ties by taking lower i
        I ← I ∪ {j} // Include set S_j into the set cover
        X ← X \ S_j // Remove elements in S_j from X.
    return I
```



Set	d(j) - 1	d(j) - 2			
S1	4	2			
S2	6				
S3	3	3			
S4	5	3			
S5	4	2			

GreedySetCover Execution (3)

/* This algorithm adds sets greedily, one at a time, until everything is covered. At each step, the algorithm chooses the next set that will cover the most uncovered elements. */

Algorithm: GreedySetCover($X, S_1, S_2, \ldots, S_m$)

Procedure:

  $I \leftarrow \emptyset$

  /* Repeat until every element in X is covered: */

  while $X \neq \emptyset$ **do**

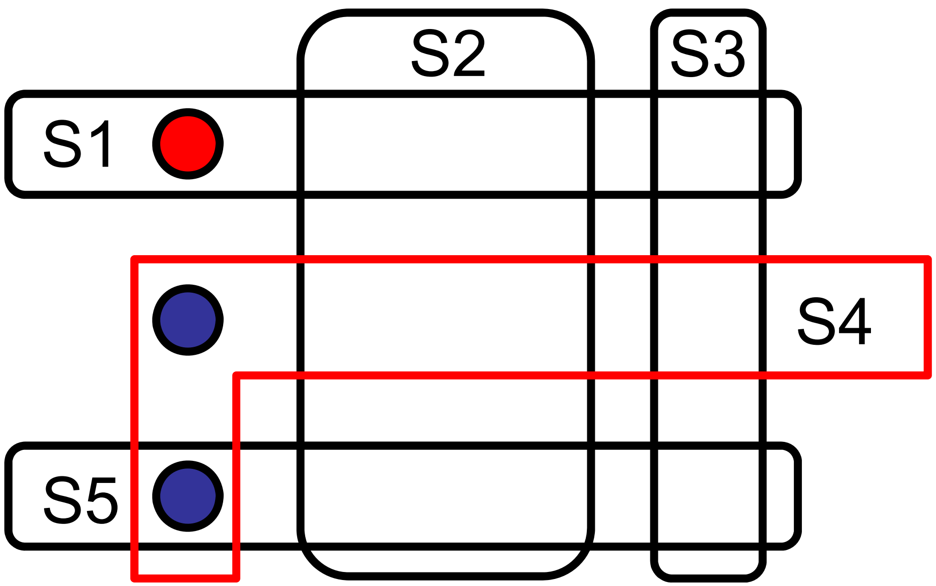
    Let $d(j) = |S_j \cap X|$ // This is the number of uncovered elements in S_j

    Let $j = \text{argmax}_{i \in \{1,2,\ldots,m\}} d(i)$ // Break ties by taking lower i

    $I \leftarrow I \cup \{j\}$ // Include set S_j into the set cover

    $X \leftarrow X \setminus S_j$ // Remove elements in S_j from X.

  return I

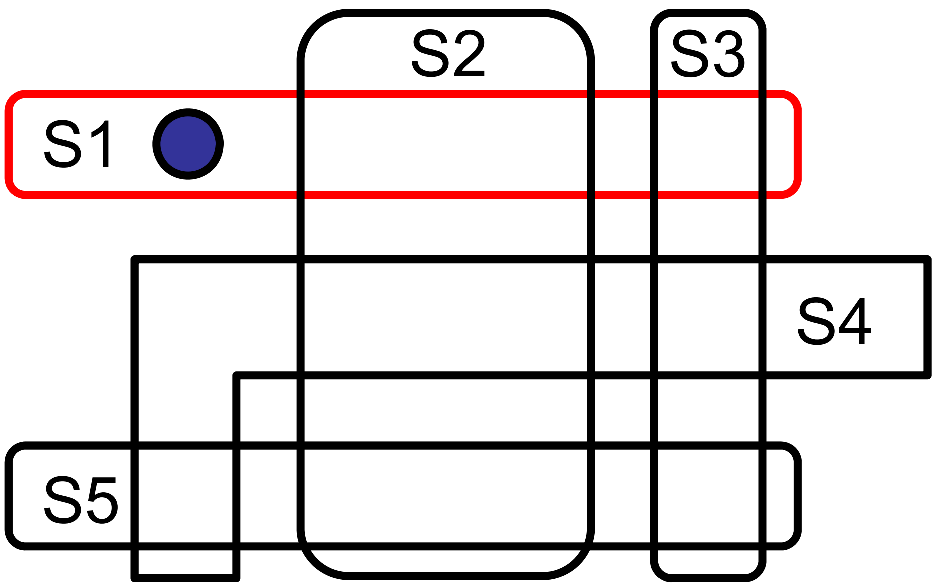


Set	d(j) - 1	d(j) - 2	d(j) - 3		
S1	4	2	1		
S2	6				
S3	3	3			
S4	5	3	2		
S5	4	2	1		

GreedySetCover Execution (4)

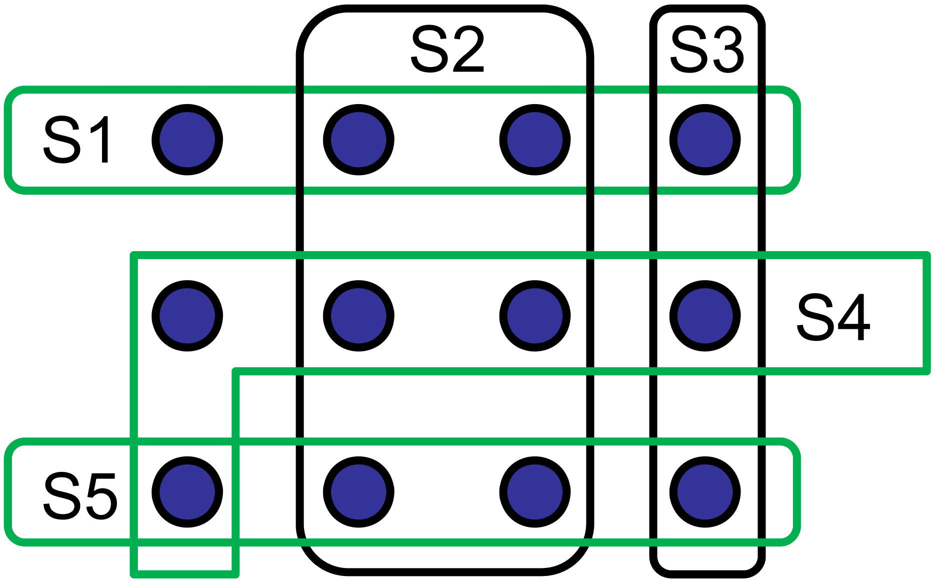
```
/* This algorithm adds sets greedily, one at a time, until everything is
   covered. At each step, the algorithm chooses the next set that will
   cover the most uncovered elements.                                    */
Algorithm: GreedySetCover(X, S_1, S_2, ..., S_m)
Procedure:
    I ← ∅
    /* Repeat until every element in X is covered:                      */
    while X ≠ ∅ do
        Let d(j) = |S_j ∩ X| // This is the number of uncovered elements in S_j
        Let j = argmax_{i∈{1,2,...,m}} d(i) // Break ties by taking lower i
        I ← I ∪ {j} // Include set S_j into the set cover
        X ← X \ S_j // Remove elements in S_j from X.
    return I
```

Time complexity: O(mn +(n+m) log m), greedy with max PQ with key = (d(i), i), the PQ must also supports **update key**

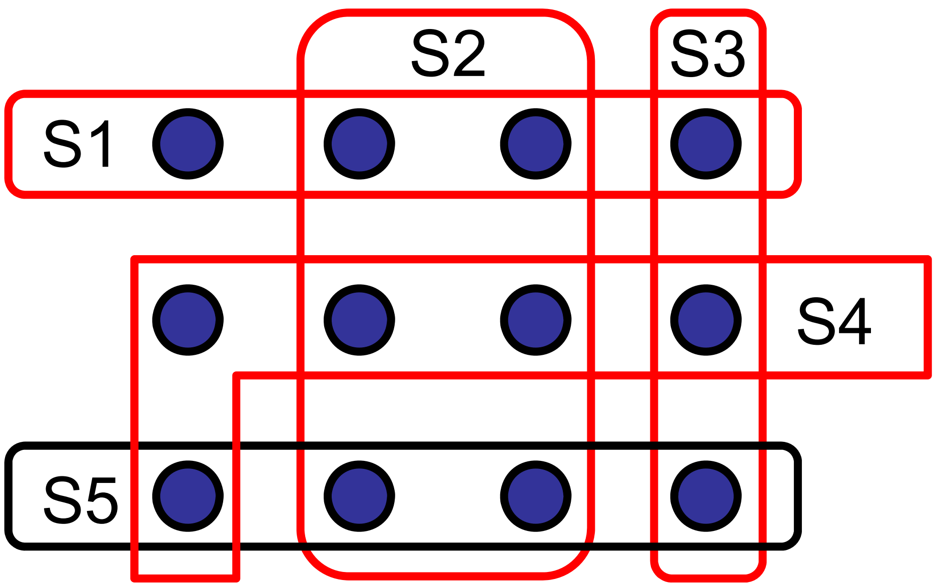


Set	d(j) - 1	d(j) - 2	d(j) - 3	d(j) - 4	
S1	4	2	1	1	
S2	6				
S3	3	3			
S4	5	3	2		
S5	4	2	1	0	

GreedySetCover vs Optimal



{S1, S4, S5} is the optimal (minimum) `Min-Set-Cover` solution for this instance



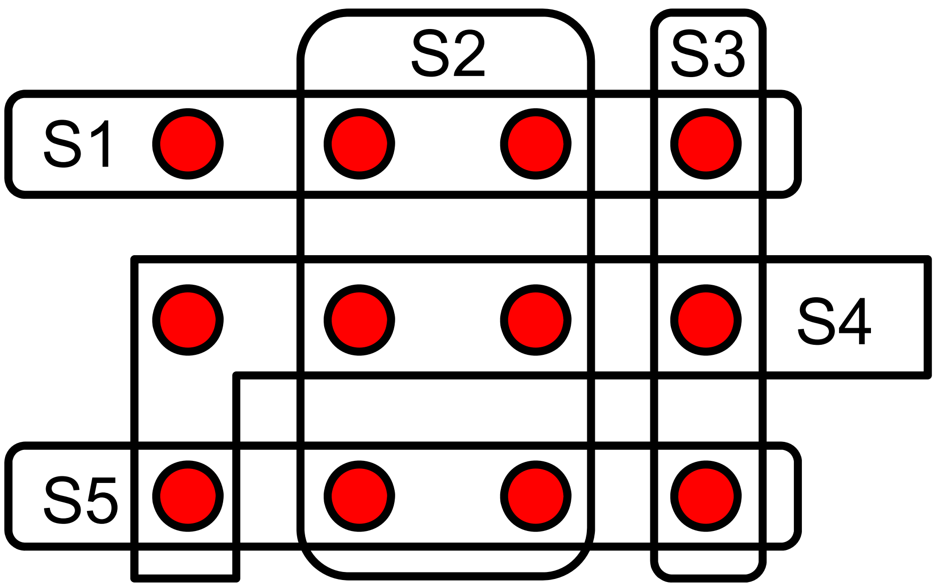
Set	d(j) - 1	d(j) - 2	d(j) - 3	d(j) - 4	
S1	4	2	1	1	
S2	6				
S3	3	3			
S4	5	3	2		
S5	4	2	1	0	

GreedySetCover – Analysis (1)

Initially, there are 12 elements

None of the sets cover more than 6 elements

Any solution for this instance of the `Min-Set-Cover` problem requires at least 12/6 = 2 sets

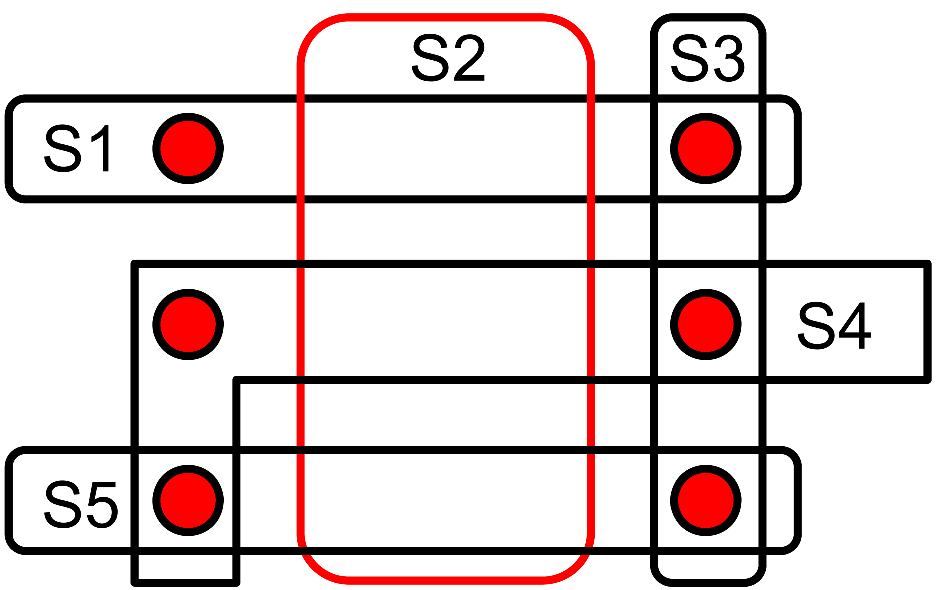


GreedySetCover – Analysis (2)

After S2 is selected greedily, there are 6 elements left

None of the available sets cover more than 3 elements

Any solution for <u>this instance</u> of the `Min-Set-Cover` problem now requires at least 6/3 = 2 (more) sets

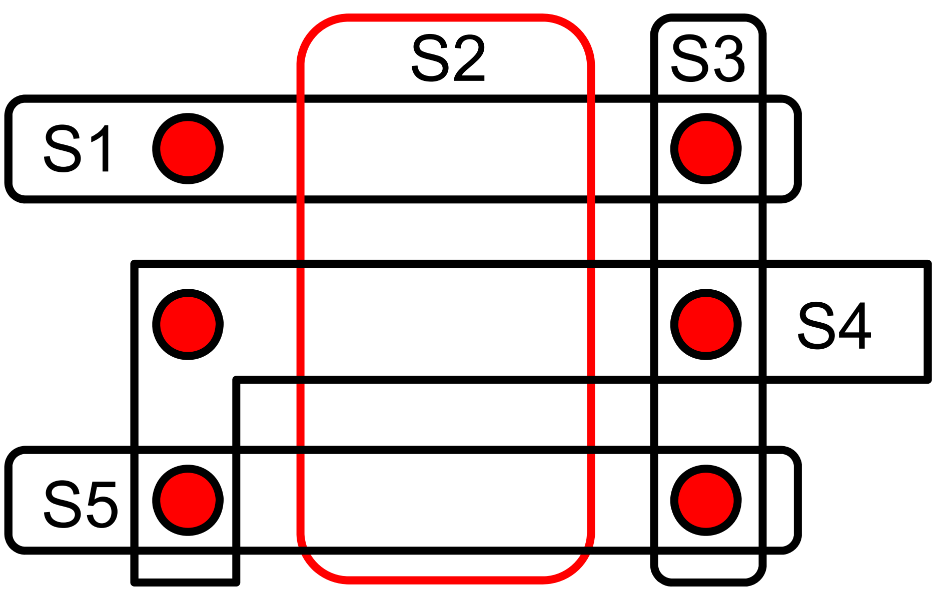


GreedySetCover – Analysis (3)

In general, if there are **k** elements left and

None of the available sets cover more than **t** elements

Any solution for <u>this instance</u> of the `Min-Set-Cover` problem now requires at least **k/t** (more) sets



GreedySetCover – Analysis (4)

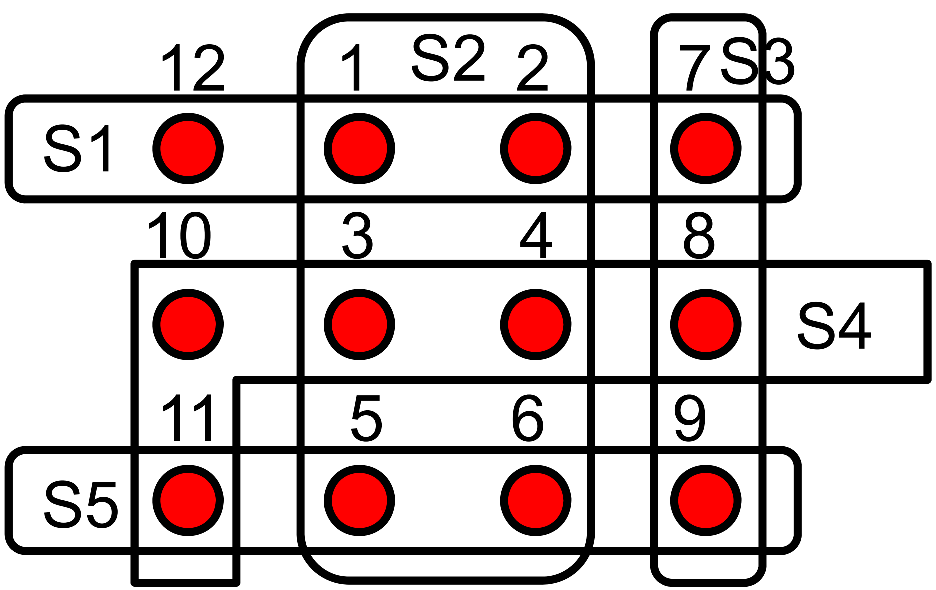
When we run the algorithm, let us label the elements in the order that they are covered.

$$\underbrace{x_1, x_2, x_3, x_4, x_5, x_6}_{S_2}, \underbrace{x_7, x_8, x_9}_{S_3}, \underbrace{x_{10}, x_{11}}_{S_4}, \underbrace{x_{12}}_{S_1}$$

These are the <u>first</u> sets that cover these elements

For each element x_j, let c_j be the number of elements covered at the same time. In the example, this would yield:

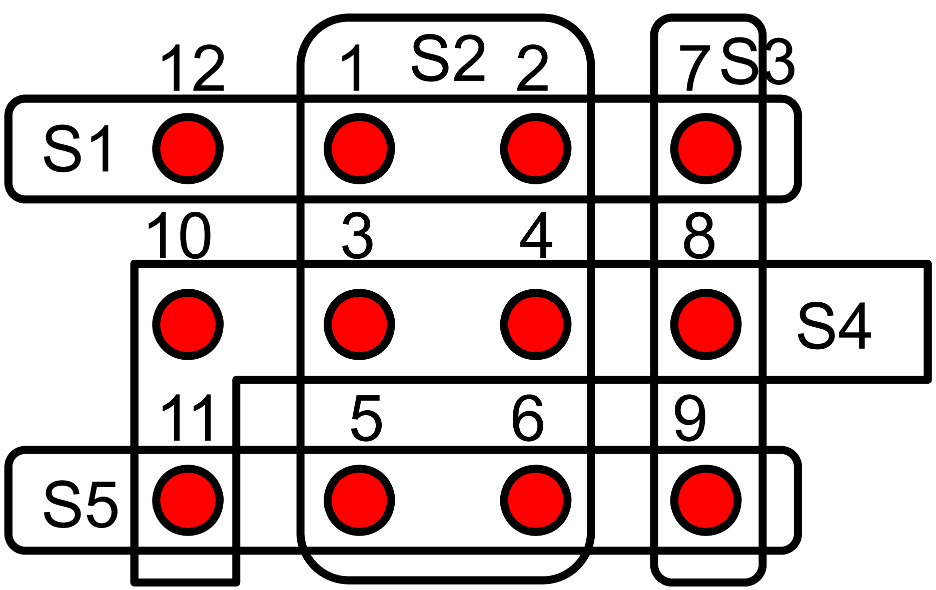
$$c_1 = 6, c_2 = 6, c_3 = 6, c_4 = 6, c_5 = 6, c_6 = 6, c_7 = 3, c_8 = 3, c_9 = 3, c_{10} = 2, c_{11} = 2, c_{12} = 1$$



GreedySetCover – Analysis (5)

We define $cost(x_j) = 1/c_j$. In this way, the cost of covering all the new elements for some set is exactly 1. In this example, the cost of covering $x_1, x_2, x_3, x_4, x_5, x_6$ is 1, the cost of covering x_7, x_8, x_9 is 1, etc. In general, if I is the set cover constructed by the Greedy Algorithm, then:

$$|I| = \sum_{j=1}^{n} cost(x_j) .$$



$cost(x_1) = 1/6$

…

$cost(x_6) = 1/6$

$cost(x_7) = 1/3$

…

$cost(x_9) = 1/3$

$cost(x_{10}) = 1/2$

$cost(x_{11}) = 1/2$

$cost(x_{12}) = 1/1$

$|I| = 4$

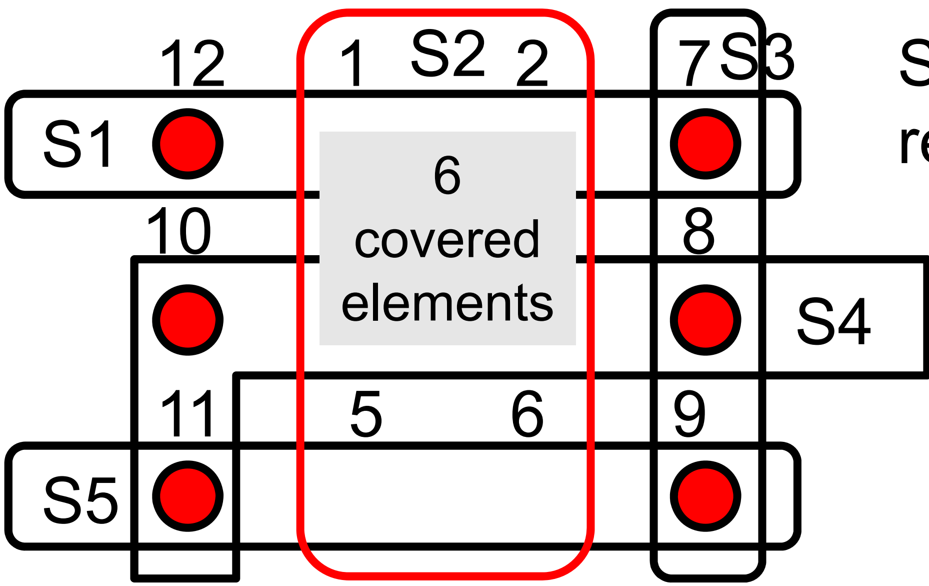
GreedySetCover – Analysis (6)

Key step. Let's consider the situation after elements $x_1, x_2, \ldots, x_{j-1}$ have already been covered, and the elements $x_j, x_{j+1}, \ldots, x_n$ remain to be covered. Let OPT be the optimal solution for covering all n elements.

What is the best that OPT can do to cover the element $x_j, x_{j+1}, \ldots, x_n$? How many sets does OPT need to cover these remaining elements?

Notice that there remain $n - j + 1$ uncovered elements. However, no set covers more than $c(j)$ of the remaining elements. In particular, all the sets already selected by the Greedy Algorithm cover *zero* of the remaining elements. Of the sets not yet chosen by the Greedy Algorithm, the one that covers the most remaining elements covers $c(j)$ of those elements: Otherwise, the Greedy Algorithm would have chosen a different set.

j=7, 12-7+1 = 6 uncovered elements



S3 covers c(7) = 3 remaining uncovered elements

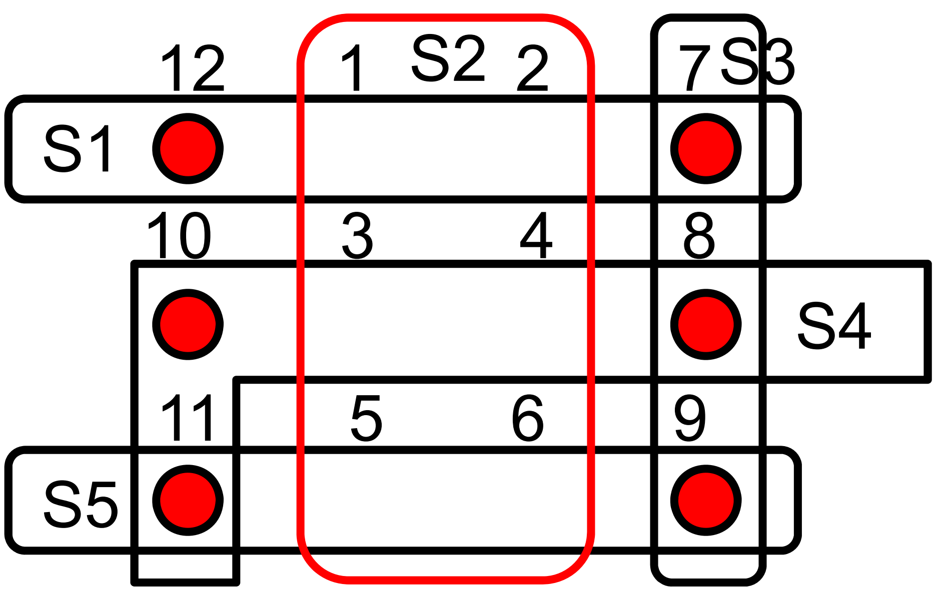
GreedySetCover – Analysis (7)

Therefore, OPT needs at least $(n-j+1)/c(j)$ sets to cover the remaining $(n-j+1)$ elements. We thus conclude that:

$$OPT \geq \frac{n-j+1}{c(j)} \geq (n-j+1)cost(x_j)$$

Or to put it differently:

$$cost(x_j) \leq \frac{OPT}{(n-j+1)}$$



GreedySetCover – Analysis (8)

We can now show that the Greedy Algorithm provides a good approximation:

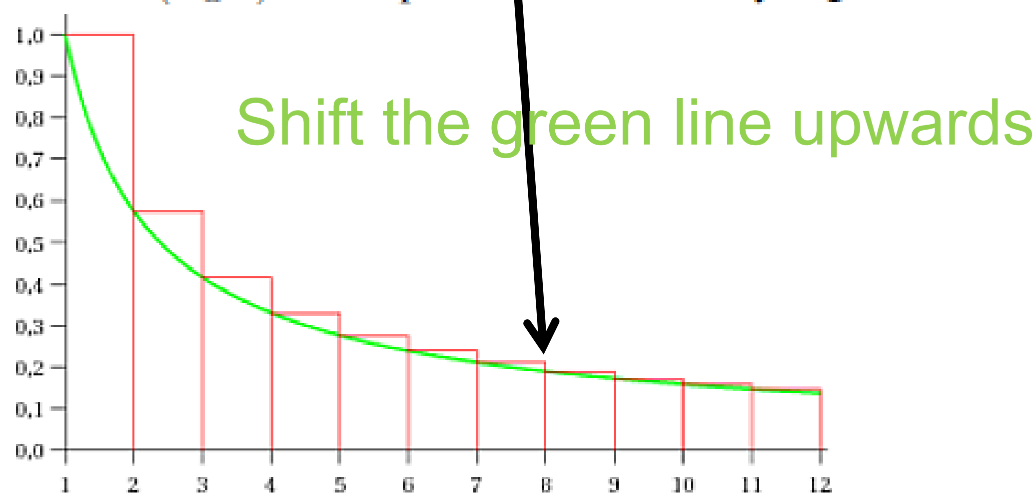
$$
\begin{aligned}
|I| &= \sum_{j=1}^{n} cost(x_j) \\
&\leq \sum_{j=1}^{n} \frac{OPT}{(n-j+1)} \\
&\leq OPT \sum_{i=1}^{n} \frac{1}{i} \\
&\leq OPT(\ln n + O(1))
\end{aligned}
$$

$$\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \ldots + \frac{1}{2} + \frac{1}{1}$$
$$=$$
$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} + \frac{1}{n}$$

(Notice that the third inequality is simply a change of variable where $i = (n - j + 1)$, and the fourth inequality is because the Harmonic series $1/1 + 1/2 + 1/3 + 1/4 + \ldots + 1/n$ can be bounded by $\ln n + O(1)$.)

We have therefore shown than the set cover constructed is at most $O(\log n)$ times optimal, i.e., the Greedy Algorithm is an $O(\log n)$-approximation algorithm:

Anecdote: And yet we said the O(log n) Deterministic-3 approximation algorithm for MVC was 'not good'… so let's debate on 'theory vs practice'



GreedySetCover – Challenge

A The Easier Ones (30 marks)

Q1. `Min-Set-Cover` Instance (6 marks)

Let's assume that we have a set X with $n = 32$. To simplify the question, assume that $X = \{1, 2, 3, \ldots, 30, 31, 32\}$. Create a `Min-Set-Cover` instance with that X and your chosen set $S = \{S_1, S_2, \ldots, S_m\}$ (The number of subsets m is up to you) so that your test case makes the $O(\ln n)$-approximation `Greedy-Set-Cover` as discussed in class produces a solution that requires $\lfloor \ln 32 \rfloor = \lfloor \log_e 32 \rfloor = \lfloor 3.46 \rfloor = 3$ times more subsets than the optimal answer.

The "easiest" question in Midterm Test S1 AY 2018/19

Review the recording for NUS students ☺

Summary

- Yet another NP-hard COP: `Min-Set-Cover`
- Four :O MSC Examples: VC, CostSaving, Tutorial, Ads
- (Optimized) complete search for small instance
- GreedySetCover (an approximation algorithm)
- Analysis: **O(log n)-**approximation algorithm
  - The proof is clever…
    - Can you do that kind of algebraic manipulation to arrive at the proof by yourself (for another analysis)?