v0.1: Seth Gilbert
V1.6: Steven Halim

CS4234
Optimiz(s)ation Algorithms

L6 - (**Weighted) Max-Cardinality** - (**Bipartite**) -**Matching, round 1**

Please read all e-Lecture slides at

https://visualgo.net/en/matching?slide=1 to 4-17

before attending this class

This course material is now made available for public usage.
Special acknowledgement to School of Computing, National University of Singapore for allowing Steven to prepare and distribute these teaching materials.

CS3233/CS4234
Dual Slides ☺

Dr. Steven Halim

Admins (to be shown live)

Roadmap (for CS4234) – Week 06

Graph Matching (this one is still PART of Midterm test)

- Overview (Quick Revision with VA)
- Unweighted MCBM:
  1. Reduction into a Max-Flow problem (quick review from T04),
  2. Augmenting Path,
  3. Dinic's or Hopcroft-Karp (nice *theoretical* result),
  4. Augmenting Path Plus
- Hall's Marriage Theorem
- **[HIDDEN SLIDES – only for NUS CS3233+CS4234 students]**
- One live demo
- A few other relevant applications will be discussed in T05 ☺
  - e.g., Matching with Capacity, Greedy Bipartite Matching (Special Case), etc

Roadmap (for CS4234) – Week 07

Graph Matching (after recess, not part of Midterm test)

- Weighted MCBM:
  1. (Min/Max) Cost (Max) Flow,
  2. Hungarian (Kuhn-Munkres algorithm)
- Unweighted MCM: Edmonds' Matching (**overview**)
- Weighted MCM: DP with Bitmask **(small graph only, review…)**
  - This DP with Bitmask solution will also solve other variants, but only if they are posed on small (V ≤ 20) graphs…
  - I am still unable to make it work for applying the Christofides' 1.5-Approximation algorithm for large instances of M-NR-TSP as of year 2022, ☹
    - PS: Now 1.5-e, see https://www.quantamagazine.org/computer-scientists-break-traveling-salesperson-record-20201008/

Graph Matching, Definitions

A matching (marriage) in a graph **G = (V, E)** (real life) is a subset **M** of **E** edges in **G** (special relationships) such that no two of which meet at a common vertex (no affair)

These definition slides are now hidden

Let’s just do a quick recap using https://visualgo.net/en/matching?slide=1

is invalid since there is an overlapping vertex

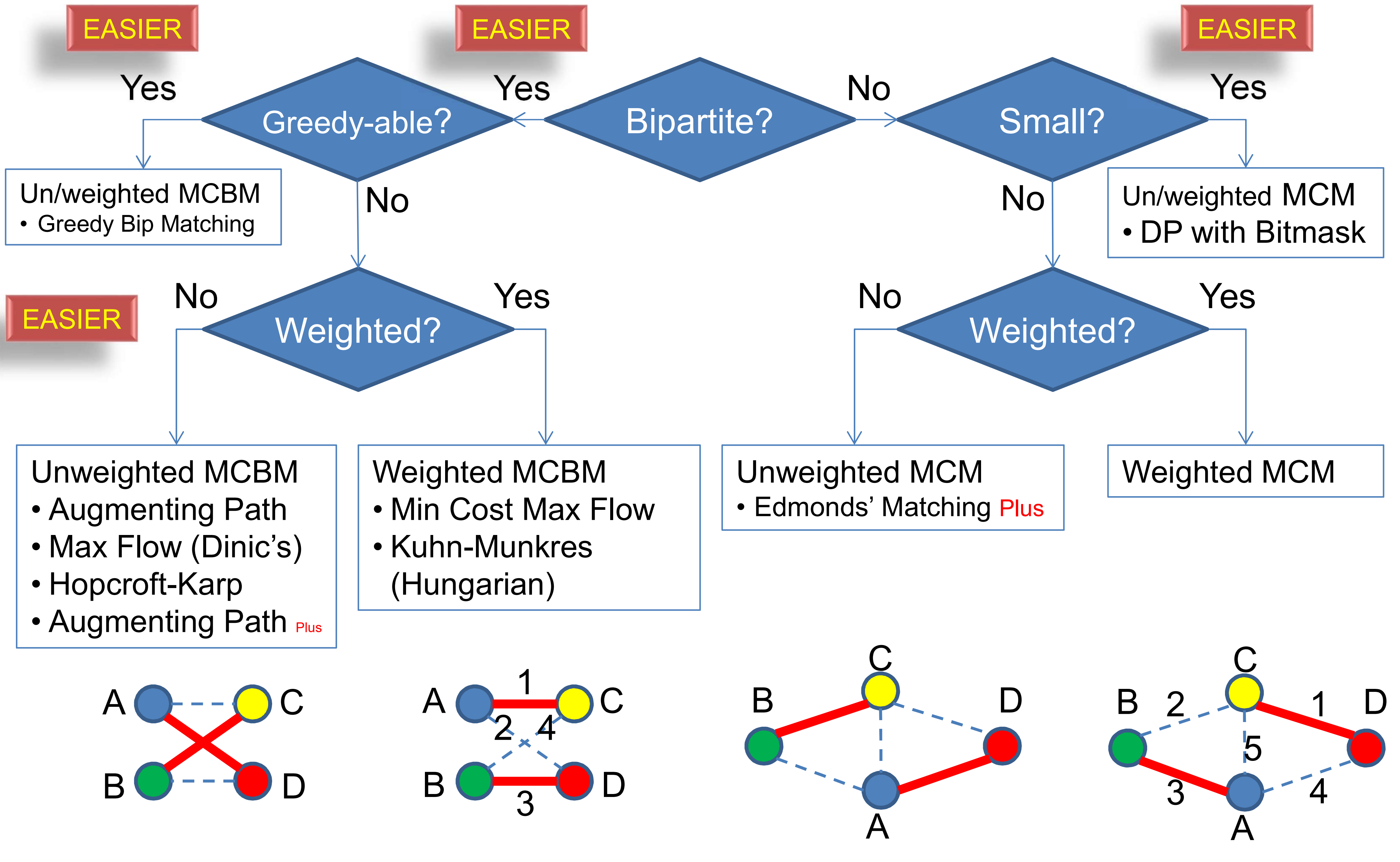
Weighted Matching

In some applications, the weights of edges are not uniform (1 unit) but varies

We may then want to take MCBM or MCM with minimum (or even maximum) total weight

[Unfortunately this is not supported in VisuAlgo ***yet***] https://www.youtube.com/watch?v=GhjwOiJ4SqU&feature=youtu.be&t=2m54s cites CP3

Types of Graph Matching



e-Lecture: https://visualgo.net/en/matching?slide=4 to slide 4-20

Solutions:

Max Flow route (longer to code, but is familiar from previous Week topic)

Augmenting Path Algorithm (very simple to implement)

Hopcroft-Karp Algorithm (essentially Dinic's Algorithm too)

Augmenting Path Algorithm Plus (with Randomized Greedy Preprocessing)

Special case: Greedy Bipartite Matching (to be discussed later in Tut05)

UNWEIGHTED MCBM

See https://visualgo.net/en/maxflow, select modeling, Bipartite Matching, all ones

Time Complexity of such modeling:
Depends on the chosen Max Flow algorithm, **but much faster** than general case as the capacities are **all ones (unit weights)** and the graph **is bipartite**, e.g., O($\mathbf{m^2}$) for basic FF, or
O(**sqrt(n)*m**) (other analysis: O(**min($\mathbf{n^{2/3}}$, $\mathbf{m^{1/2}}$)*m**)) for Dinic's
(these have been discussed briefly in Tut04)

e-Lecture: https://visualgo.net/en/matching?slide=4-1 and 4-2

Finding MCBM by reducing this problem into

MAX FLOW
CP4 BOOK 2 SECTION 8.4 & 8.5

CP4 Book 2 Ex 8.5.3.1*: Why the graph must be directed?

- Answer hidden, review the recording

e-Lecture: https://visualgo.net/en/matching?slide=4-3 to slide 4-12

Finding MCBM via

AUGMENTING PATH ALGORITHM CP4 BOOK 1 SECTION 4.6.3

Berge's Theorem (previously lemma)

- In 1957, a French mathematician Claude Berge proves his (Berge's) theorem:
  - "a matching **M** in a graph G is maximum *if and only if* there is **no augmenting path** with **M**".
  - Augmenting path is a path that starts and ends on free (unmatched) vertices, and alternates between edges in and not in the matching
- We will quickly show the proof
  - We will revisit this proof in Tut05 in case the presentation is too fast
- Then, we will directly use the result of this theorem: write an algorithm that finds augmenting path, and eliminates it

e-Lecture: https://visualgo.net/en/matching?slide=4-13 to slide 4-14

Finding MCBM via

HOPCROFT-KARP (DINIC'S) ALGORITHM
CP4 BOOK 2 SECTION 9.26

https://github.com/stevenhalim/cpbook-code/blob/master/ch4/mcbm.cpp (or py/java/ml)

e-Lecture: https://visualgo.net/en/matching?slide=4-15 to slide 4-20

Finding MCBM via

AUGMENTING PATH ALGORITHM PLUS (WITH RANDOMIZED GREEDY PREPROCESSING) CP4 BOOK 2 SECTION 8.5.3

When To Use Max Flow Route?

To be discussed live

When To NOT Use MCBM Algorithm at all?

One task reserved for Tut05 ☺

Hall's Marriage Theorem

- Let **G** be a bipartite graph with bipartite sets **U** and **V**
- Then there exists a matching that ***covers U***
  if and only if ***for each subset W*** of **U**, **|W| ≤ |N(W)|**
- Notice that "for each subset" means $2^{|W|}$ checks
  - Only applicable if $2^{|W|}$ is "small enough"

When we return: MCBM applications…
including special cases of some NP-hard problems…

Please read https://nus.kattis.com/problems/bookcircle first

LECTURE BREAK

PROBABLY THE MOST IMPORTANT PART OF THIS LECTURE

SO KEEP AN EYE ON LEFT/RIGHT, ROW/COL, alternate ROW/COL, PRIME/COPRIME, ODD/EVEN, MALE/FEMALE, JOB/EMPLOYEE, bi-coloring, out-degree only/in-degree only, no-odd-length-cycle, Tree*, etc (and lots of other common stuffs)
(see CP4 Exercise 4.6.3.1*)

MORE EXAMPLES OF MCBM PROBLEMS

https://visualgo.net/en/matching, see modeling, RookAttack

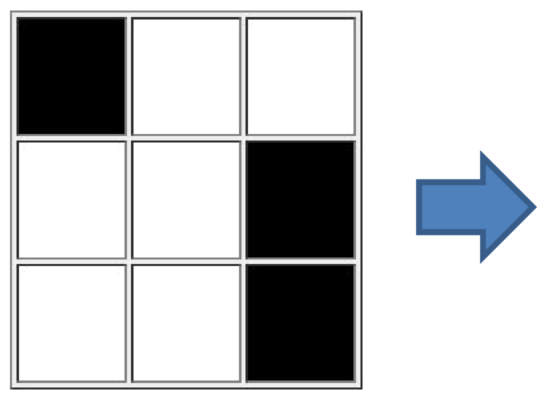
RookAttack (TopCoder)

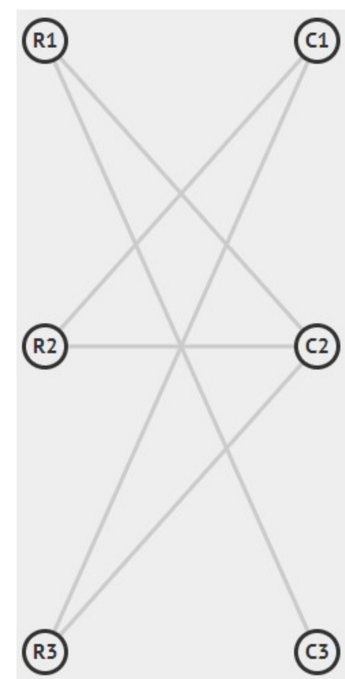
Problem description (abridged):

- Place a max number of non-attacking rooks on a *rows x cols* board with some squares cut out
  - Rooks attack pieces in the same row and/or col
- Cut out squares do not affect where the rooks can attack

Solution: Reduce to MCBM

- Set A: rows, Set B: cols
- Edge: if cell(row **i**, col **j**) is OK, link **i** in set A to **j** in set B
  - If row **i** is matched with col **j**, both row i and col j cannot be matched with anything else
    - Maximizes the MCBM, this is what we want



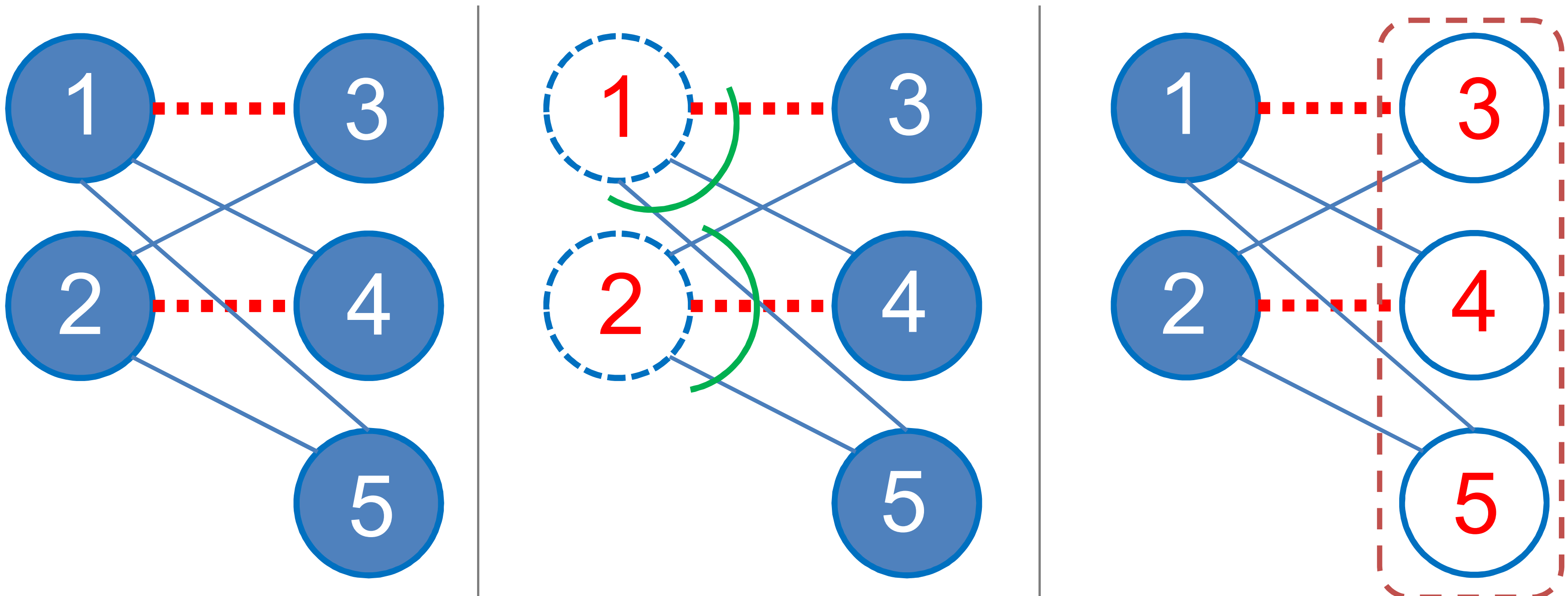


Popular Variants

Min(imum) Vertex Cover / Max(imum) Independent Set

CP4 Book 2 Section 8.6.6

These are **NP-Hard optimization problems** on general graph, see https://visualgo.net/en/mvc (also the first two weeks of CS4234)



A. MCBM

B. Min Vertex Cover
|MVC| = |MCBM|

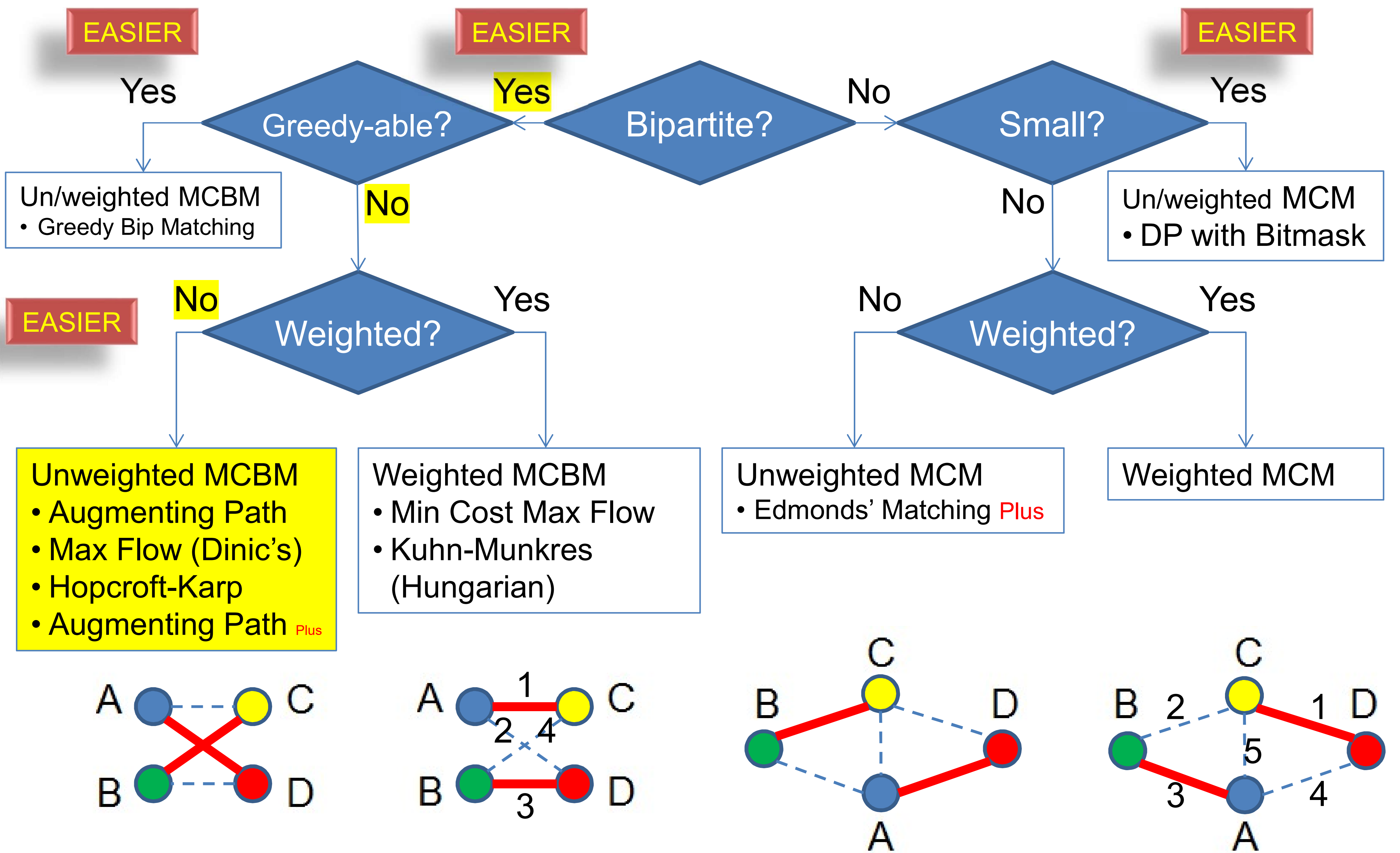
C. Max Independent Set
|MIS| = |V| – |MCBM|

(König's theorem)
constructive proof in Tut05

Live solve [an MCBM-related problem]

- Let's discuss the MCBM modeling live
- Review the recording

Summary (so far...)



References

- Mostly CP4, these sections:
  - 3.4 (Greedy bipartite matching special case)
  - 4.6.3 (MCBM as specific algorithm for a special graph: Bipartite Graph),
  - 8.5 (Graph Matching overview),
  - 8.6.6 (MIS/MVC on Bipartite)
  - 8.6.8 (MPC on DAG),
  - 9.25 (Min Cost (Max) Flow)
  - 9.26 (Hopcroft-Karp Algorithm),
  - 9.27 (Kuhn-Munkres Algorithm),
  - 9.28 (Edmonds' Matching Algorithm), and
  - 9.29 (Chinese Postman Problem),
- TopCoder PrimePairs, RookAttack solution
- http://www.comp.nus.edu.sg/~cs6234/2009/Lectures/Match-sl-PC.pdf (Prof LeongHW's/P Karras slides)

Enjoy your recess week break ☺

- Unless you still fighting with PS3 >.<
  or want to do PS4 early :O…
  – At least |half recess break| then… 2-OPT :O