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Special acknowledgement to School of Computing, National University of Singapore for allowing Steven to prepare and distribute these teaching materials.

CS3233
Competitive Programming
Dr. Steven Halim
Week 04 – Problem Solving Paradigms
(Dynamic Programming 1)
Outline

• Mini Contest #3 + Break + Discussion + Admins
• Dynamic Programming – Introduction
  – Treat this as revision for ex CS2010/CS2020 students
  – Listen carefully for the other group of students!
  – I open consultation slots (Mon/Fri) for NUS students who need help with this topic, especially those who did not go through CS2010/CS2020 before
• Dynamic Programming
  – Some Classical Examples
• PS: I will use the term DP in this lecture
  – OOT: DP is NOT Down Payment!
Wedding Shopping

EXAMPLE 1
Motivation

• How to solve UVa 11450 (Wedding Shopping)?
  – Given $1 \leq C \leq 20$ classes of garments
    • e.g. shirt, belt, shoe
  – Given $1 \leq K \leq 20$ different models for each class of garment
    • e.g. three shirts, two belts, four shoes, ..., each with its own price
  – Task: Buy just one model of each class of garment
  – Our budget $1 \leq M \leq 200$ is limited
    • We cannot spend more money than it
    • But we want to spend the maximum possible
  – What is our maximum possible spending?
    • Output “no solution” if this is impossible
• Budget M = 100
  – Answer: 75

• Budget M = 20
  – Answer: 19
  • Alternative answers are possible

• Budget M = 5
  – Answer: no solution
Greedy Solution?

• What if we buy the most expensive model for each garment which still fits our budget?

• Counter example:
  – \( M = 12 \)
  – Greedy will produce:
    • no solution
  – Wrong answer!
    • The correct answer is 12
    • (see the green dotted highlights)
    – Q: Can you spot one more potential optimal solution?

<table>
<thead>
<tr>
<th>Model Garment</th>
<th>0</th>
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<td>C = 2</td>
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</table>
Divide and Conquer?

• Any idea?
Complete Search? (1)

• What is the potential **state** of the problem?
  – g (which garment?)
  – id (which model?)
  – money (money left?)

• Answer:
  – (money, g) or (g, money)

• Recurrence (recursive backtracking function):

```python
def shop(money, g):
    if money < 0:
        return -INF
    if g == C:
        return M - money
    return max(shop(money - price[g][model], g + 1), \forall model \in [1..K])
```
Complete Search? (2)

• But, how to solve this?
  – $M = 200$ (maximum)
• Time Complexity: $20^{20}$
  • Too many for $3s$ time limit 😞

<table>
<thead>
<tr>
<th>Model</th>
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<th>...</th>
<th>19</th>
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<td>99</td>
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Overlapping Sub Problem Issue

- In the simple $20^{20}$ Complete Search solution, we observe many overlapping sub problems!
  - Many ways to reach state (money, g), e.g. see below, $M = 12$

<table>
<thead>
<tr>
<th>Model Garment</th>
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<tr>
<td>C = 2</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>
DP to the Rescue (1)

• DP = Dynamic Programming
  • Programming here is not writing computer code, but a “tabular method”!
    – a.k.a. table method
  • A programming paradigm that you must know!
    – And hopefully, master...
DP to the Rescue (2)

• Use DP when the problem exhibits:
  – Optimal sub structure
    • Optimal solution to the original problem contains optimal solution to sub problems
      – This is similar as the requirement of Greedy algorithm
      – If you can formulate complete search recurrences, you have this
  – Overlapping sub problems
    • Number of distinct sub problems are actually “small”
    • But they are repeatedly computed
      – This is different from Divide and Conquer
DP Solution – Implementation (1)

• There are two ways to implement DP:
  – Top-Down
  – Bottom-Up

• Top-Down (Demo):
  – Recursion as per normal + memoization table
    • It is just a simple change from backtracking (complete search) solution!
Turn Recursion into Memoization

initialize memo table in main function (use ‘memset’)

return_value recursion(params/state) {
  if this state is already calculated,
    simply return the result from the memo table
  calculate the result using recursion(other_params/states)
  save the result of this state in the memo table
  return the result
}
Dynamic Programming (Top-Down)

• For our example:

```java
shop(money, g)
    if (money < 0) return -INF
    if (g == C) return M - money
    if (memo[money][g] != -1) return memo[money][g];
    return memo[money][g] = max(shop(money - price[g][model], g + 1),
        ∀model ∈ [1..K])
```

• As simple as that 😊
If Optimal Solution(s) are Needed

• Clever solution for Top-Down DP
  — (See solution for Bottom-Up DP in Example 2)

• For our example:

```python
print_shop(money, g)
    if (money < 0 || g == C) return
    for each model ∈ [1..K]
        if shop(money - price[g][model], g + 1) == memo[money][g]
            print "take model = " + model + " for garment g = " + g
            print_shop(money - price[g][model], g + 1)
            break
```

• As simple as that 😊
DP Solution – Implementation (2)

• Another way: Bottom-Up:
  – Prepare a table that has size equals to the number of distinct states of the problem
  – Start to fill in the table with base case values
  – Get the topological order in which the table is filled
    • Some topological orders are natural and can be written with just (nested) loops!
  – Different way of thinking compared to Top-Down DP

• Notice that both DP variants use “table”!
Dynamic Programming (Bottom-Up)

• For our example:
  – Start with table `can_reach` of size 20 (g) * 201 (money)
    • The state (money, g) is reversed to (g, money) so that we can process bottom-up DP loops in row major fashion
    • Initialize all entries to 0 (false)
    • Fill in the first row with money left (column) reachable after buying models from the first garment (g = 0)
  – Use the information of current row g to update the values at the next row g + 1
- Budget $M = 20$
  - Answer: 19
- Alternative answers are possible

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<td>$C = 2$</td>
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money =>

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Top-Down or Bottom-Up?

- **Top-Down**
  - **Pro:**
    - Natural transformation from normal recursion
    - Only compute sub problems when necessary
  - **Cons:**
    - Slower if there are many sub problems due to recursive call overhead
    - Use exactly \(O(\text{states})\) table size (MLE?)

- **Bottom Up**
  - **Pro:**
    - Faster if many sub problems are visited: no recursive calls!
    - Can save memory space*
  - **Cons:**
    - Maybe not intuitive for those inclined to recursions?
    - If there are \(X\) states, bottom up visits/fills the value of all these \(X\) states
Flight Planner
(study this on your own)

EXAMPLE 2
Motivation

• How to solve this: 10337 (Flight Planner)?
  
  – Unit: 1 mile altitude and 1 (x100) miles distance
  
  – Given wind speed map
  
  – Fuel cost: {climb (+60), hold (+30), sink (+20)} - wind speed wsp[alt][dis]
  
  – Compute min fuel cost from (0, 0) to (0, X = 4)!

```
1 1 1 1 | 9
1 1 1 1 | 8
1 1 1 1 | 7
1 1 1 1 | 6
1 1 1 1 | 5
1 1 1 1 | 4
1 1 1 1 | 3
1 1 1 1 | 2
1 9 9 1 | 1
1 -9 -9 1 | 0
```

```
0 1 2 3 4 (x100)
```
Complete Search? (1)

• First guess:
  – Do complete search/brute force/backtracking
  – Find *all possible* flight paths and pick the one that yield the minimum fuel cost
Complete Search? (2)

- Recurrence of the Complete Search
  - \( \text{fuel}(\text{alt}, \text{dis}) = \)
    \[
    \min_3 (60 - \text{wsp[alt][dis]} + \text{fuel}(\text{alt + 1}, \text{dis + 1}),
    30 - \text{wsp[alt][dis]} + \text{fuel}(\text{alt}, \text{dis + 1}),
    20 - \text{wsp[alt][dis]} + \text{fuel}(\text{alt - 1}, \text{dis + 1}))
    \]
  - Stop when we reach final state (base case):
    - \( \text{alt} = 0 \) and \( \text{dis} = X \), i.e. \( \text{fuel}(0, X) = 0 \)
  - Prune infeasible states (also base cases):
    - \( \text{alt} < 0 \) or \( \text{alt} > 9 \) or \( \text{dis} > X! \), i.e. return \( \text{INF}^* \)
  - Answer of the problem is \( \text{fuel}(0, 0) \)
Complete Search Solutions (1)

• Solution 1

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 | 9 \\
1 & 1 & 1 & 1 | 8 \\
1 & 1 & 1 & 1 | 7 \\
1 & 1 & 1 & 1 | 6 \\
1 & 1 & 1 & 1 | 5 \\
1 & 1 & 1 & 1 | 4 \\
1 & 1 & 1 & 1 | 3 \\
1 & 1 & 1 & 1 | 2 \\
1 & 9 & 9 & 1 | 1 \\
1 & 9 & 9 & 1 | 0 \\
0 & 1 & 2 & 3 & 4 (x100)
\end{array}
\]

29+ 39+ 39+ 29=136

• Solution 2

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 | 9 \\
1 & 1 & 1 & 1 | 8 \\
1 & 1 & 1 & 1 | 7 \\
1 & 1 & 1 & 1 | 6 \\
1 & 1 & 1 & 1 | 5 \\
1 & 1 & 1 & 1 | 4 \\
1 & 1 & 1 & 1 | 3 \\
1 & 1 & 1 & 1 | 2 \\
1 & 9 & 9 & 1 | 1 \\
1 & 9 & 9 & 1 | 0 \\
0 & 1 & 2 & 3 & 4 (x100)
\end{array}
\]

29+ 39+ 69+ 19=156
Complete Search Solutions (2)

- **Solution 3**

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29 + 69 + 11 + 29 = 138

- **Solution 4**

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</tr>
</thead>
</table>

59 + 11 + 39 + 29 = 138

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Complete Search Solutions (3)

- **Solution 5**

  | 1 | 1 | 1 | 1 |   |
  | 9 |
  | 1 | 1 | 1 | 1 |   |
  | 8 |
  | 1 | 1 | 1 | 1 |   |
  | 7 |
  | 1 | 1 | 1 | 1 |   |
  | 6 |
  | 1 | 1 | 1 | 1 |   |
  | 5 |
  | 1 | 1 | 1 | 1 |   |
  | 4 |
  | 1 | 1 | 1 | 1 |   |
  | 3 |
  | 1 | 1 | 1 | 1 |   |
  | 2 |
  | 1 | 9 | 9 | 1 |   |
  | 1 |
  | 1 | -9 | -9 | 1 |   |
  | 0 |

  0   1   2   3   4  (x100)

  29+ 69+ 21+ 19=138

- **Solution 6**

  | 1 | 1 | 1 | 1 |   |
  | 9 |
  | 1 | 1 | 1 | 1 |   |
  | 8 |
  | 1 | 1 | 1 | 1 |   |
  | 7 |
  | 1 | 1 | 1 | 1 |   |
  | 6 |
  | 1 | 1 | 1 | 1 |   |
  | 5 |
  | 1 | 1 | 1 | 1 |   |
  | 4 |
  | 1 | 1 | 1 | 1 |   |
  | 3 |
  | 1 | 1 | 1 | 1 |   |
  | 2 |
  | 1 | 9 | 9 | 1 |   |
  | 1 |
  | 1 | -9 | -9 | 1 |   |
  | 0 |

  0   1   2   3   4  (x100)

  59+ 21+ 11+ 29=120 (OPT)
Complete Search Solutions (4)

• Solution 7

\[
\begin{array}{cccc|c}
1 & 1 & 1 & 1 & 9 \\
1 & 1 & 1 & 1 & 8 \\
1 & 1 & 1 & 1 & 7 \\
1 & 1 & 1 & 1 & 6 \\
1 & 1 & 1 & 1 & 5 \\
1 & 1 & 1 & 1 & 4 \\
1 & 1 & 1 & 1 & 3 \\
1 & 1 & 1 & 1 & 2 \\
1 & -9 & -9 & 1 & 1 \\
1 & 9 & 9 & 1 & 0 \\
\end{array}
\]

\[
0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \text{(x100)}
\]

59 + 21 + 21 + 19 = 120 (OPT)

• Solution 8

\[
\begin{array}{cccc|c}
1 & 1 & 1 & 1 & 9 \\
1 & 1 & 1 & 1 & 8 \\
1 & 1 & 1 & 1 & 7 \\
1 & 1 & 1 & 1 & 6 \\
1 & 1 & 1 & 1 & 5 \\
1 & 1 & 1 & 1 & 4 \\
1 & 1 & 1 & 1 & 3 \\
1 & 1 & 1 & 1 & 2 \\
1 & 9 & 9 & 1 & 1 \\
1 & -9 & -9 & 1 & 0 \\
\end{array}
\]

\[
0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \text{(x100)}
\]

59 + 51 + 19 + 19 = 148
Complete Search? (3)

• How large is the search space?
  – Max distance is 100,000 miles
    Each distance step is 100 miles
    That means we have $1,000$ distance columns!
    • Note: this is an example of “coordinate compression”
  – Branching factor per step is 3... (climb, hold, sink)
  – That means complete search can end up performing $3^{1,000}$ operations...
  – Too many for 3s time limit 😞
Overlapping Sub Problem Issue

• In simple $3^{1,000}$ Complete Search solution, we observe many overlapping sub problems!
  – Many ways to reach coordinate (alt, dis)
DP Solution

• Recurrence* of the Complete Search
  – $\text{fuel}(\text{alt}, \text{dis}) =$
    \[
    \min\{60 - \text{wsp}[\text{alt}][\text{dis}] + \text{fuel}(\text{alt} + 1, \text{dis} + 1), \\
    30 - \text{wsp}[\text{alt}][\text{dis}] + \text{fuel}(\text{alt}, \text{dis} + 1), \\
    20 - \text{wsp}[\text{alt}][\text{dis}] + \text{fuel}(\text{alt} - 1, \text{dis} + 1)\}
    \]

• Sub-problem fuel(alt, dis) can be overlapping!
  – There are only 10 alt and 1,000 dis = 10,000 states
  – A lot of time saved if these are not re-computed!
    • Exponential $3^{1,000}$ to polynomial $10 \times 1,000$!
DP Solution (Top Down)

- Create a 2-D table of size 10 * (X/100)
  - Set “-1” for unexplored sub problems (memset)
  - Store the computation value of sub problem

- Simply reuse when it is needed again!

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 \\
0 & -1 & -1 & -1 & 29 & 0 \\
1 & -1 & -1 & 40 & 19 & \infty \\
2 & -1 & -1 & \infty & \infty & \\
\end{array}
\]
## DP Solution (Bottom Up)

**fuel**\( (\text{alt}, \text{dis}) \) =

\[
\min(20 - \text{wsp}[\text{alt} + 1][\text{dis} - 1] + \text{fuel}(\text{alt} + 1, \text{dis} - 1), \\
30 - \text{wsp}[\text{alt}][\text{dis} - 1] + \text{fuel}(\text{alt}, \text{dis} - 1), \\
60 - \text{wsp}[\text{alt} - 1][\text{dis} - 1] + \text{fuel}(\text{alt} - 1, \text{dis} - 1))
\]

### Tips:
**space-saving trick**

We can reduce one storage dimension by only keeping 2 recent columns at a time...

But the time complexity is unchanged: \(O(10 \times X / 100)\)

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</tr>
</tbody>
</table>

\(0 \ 1 \ 2 \ 3 \ 4 \ (x100)\)
If Optimal Solution(s) are Needed

- Although not often, sometimes this is asked!
- As we build the DP table, record which option is taken in each cell!
  - Usually, this information is stored in different table
  - Then, do recursive scan(s) to output solution
    - Sometimes, there are more than one solutions!

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>∞</td>
<td>∞</td>
<td>110</td>
<td>131</td>
</tr>
<tr>
<td>1</td>
<td>∞</td>
<td>59</td>
<td>80</td>
<td>101</td>
</tr>
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<td>0</td>
<td>29</td>
<td>68</td>
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<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>
Shortest Path Problem? (1)

• Hey, I have alternative solution:
  – Model the problem as a DAG
  – Vertex is each position in the unit map
  – Edges connect vertices reachable from vertex (alt, dis), i.e. (alt+1, dis+1), (alt, dis+1), (alt-1, dis)
    • Weighted according to flight action and wind speed!
    • Do not connect infeasible vertices
      – alt < 0 or alt > 9 or dis > X
Visualization of the DAG

What is the shortest path from source to destination?
Shortest Path Problem? (2)

- The problem: find the *shortest path* from vertex (0, 0) to vertex (0, X) on this DAG...
- \(O(V + E)\) solution exists!
  - \(V\) is just \(10 \times (X / 100)\)
  - \(E\) is just \(3V\)
  - Thus this solution is as good as the DP solution
Break

• Coming up next, discussion of some **Classical DPs**:
  – Max Sum (1-D for now) → Kadane’s Algorithm
  – Longest Increasing Subsequence (LIS) → O(n log k) solution
  – 0-1 Knapsack / Subset Sum → Knapsack-style parameter!
  – Coin Change (the General Case) → skipped, see textbook
  – Traveling Salesman Problem (TSP) → bitmask again :O

• I will try to cover as many as possible, but will stop at 9 pm 😊; the details are in Chapter 3 of CP2.9
Let’s discuss several problems that are solvable using DP
First, let’s see some classical ones...

LEARNING VIA EXAMPLES
Max Sum (1D)

• Find a **contiguous sub-array** in 1D array A with the **max sum**

<table>
<thead>
<tr>
<th>i</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>A[i]</td>
<td>1</td>
<td>-2</td>
<td>6</td>
<td>3</td>
<td>2</td>
<td>-12</td>
<td>-6</td>
<td>7</td>
<td>1</td>
</tr>
</tbody>
</table>

• The answer is \(\{6, 3, 2\}\) with max sum \(6 + 3 + 2 = 11\)

Can we do this in \(O(n^3)\)?
Can we do this in \(O(n^2)\)?
Can we do this in \(O(n)\)?

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Longest Increasing Subsequence

- Find the Longest Increasing **Subsequence** (LIS) in array A
  - Subsequence is not necessarily contiguously

<table>
<thead>
<tr>
<th>i</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>A[i]</td>
<td>-7</td>
<td>10</td>
<td>9</td>
<td>2</td>
<td>3</td>
<td>8</td>
<td>8</td>
<td>1</td>
</tr>
</tbody>
</table>

- The answer is {-7, 2, 3, 8} with length 4

Can we do this in $O(n^2)$?

Can we do this in $O(n \log k)$?
0-1 Knapsack / Subset Sum

Can we do this in O(nS)?

Red = 15 kg, $7
Blue = 8 kg, $15

n = # items
S = knapsack size

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Traveling Salesman Problem (TSP)

<table>
<thead>
<tr>
<th>dist</th>
<th>0</th>
<th>1</th>
<th>...</th>
<th>n-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>...</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>n-1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Traveling Salesman Problem (TSP)

- **State:** \( \text{tsp}(\text{pos, bitmask}) \)
- **Transition:**
  - If every cities have been visited
    - \( \text{tsp}(\text{pos, } 2^N-1) = \text{dist}[\text{pos}][0] \)
  - Else, try visiting unvisited cities one by one
    - \( \text{tsp}(\text{pos, bitmask}) = \min(\text{dist}[\text{pos}][\text{nxt}] + \text{tsp}(\text{nxt, bitmask | (1 << nxt)))} \)
    \( \forall \text{nxt} \in [0..N-1], \text{nxt} \neq \text{pos, bitmask} \& (1 << \text{nxt}) = 0 \)
Summary

• We have seen:
  – Basic DP concepts
  – DP on some **classical** problems

• We will see more DP next week:
  – DP on **non classical** problems
  – DP and its relationship with DAG
  – DP on Math & String Problems
  – Some other “cool” DP (optimization) techniques
Good References about DP

• CP2.9, obviously 😊
  – Section 3.5 first
    • Then Section 4.7.1 (DAG), 5.4 (Combinatorics), 6.5 (String + DP), 8.3 (more advanced DP), parts of Ch 9

• http://people.csail.mit.edu/bdean/6.046/dp/
  – Current USACO Director

• TopCoder Algorithm Tutorial
  – http://community.topcoder.com/tc?module=Static&d1=tutorials&d2=dynProg