Proving Inequalities Elegantly using Sum of Squares Technique

Anh Cuong Nguyen
Department of Computer Science – National University of Singapore
anhcuong@comp.nus.edu.sg

1 Introduction

Inequality is an interesting and difficult topic in Mathematics. Typically, solving an inequality involve reducing the problem to some well-known inequalities such as AMV or GMV[1]. Difficult inequalities require more complex transformations before they can be reduced into well-known inequalities (e.g. IRAN'96 inequality requires total expansion to reduce to SCHUR inequality). This strategy is however rather ad-hoc and usually produce complicated solutions. In this article we introduce a novel technique for proving inequalities which involve only a few systematic transformation steps. We name the technique S.O.S (SUM OF SQUARES) and find that the technique can produce elegant solutions for many *previously-known-complicated* inequalities.

2 Preliminary Idea

To get the sense of S.O.S idea, firstly consider a series of some classic inequalities, such as CAUCHY, HOLDER, BERNOULLI ... and observe that most inequalities can be reduced into CAUCHY inequality. Indeed CAUCHY inequality can be considered as the *origin* of many classic inequalities. So does CAUCHY inequality have some origins to which it can be further reduced too? We deem to say that the answer is yes, and the origin of CAUCHY is $x^2 \geq 0$, or more concretely, $(a-b)^2 \geq 0$. So could all inequalities, which are constructed from two variables, and have the equality occurs when the two variables are equal, be transformed into the form $(a-b)^2$, consequently, the transformation is sufficient to prove these inequalities? The answer is, again, yes and this observation leads to a powerful technique for proving inequalities systematically and elegantly that we call SUM OF SQUARES, or S.O.S.

We will first familiarize with the technique to transform a two variables inequality to the form $g(a,b)(a-b)^2$, which we call S.O.S form of two variables.. When the transformation is done, it is sufficient to prove that $g(a,b) \geq 0$ to prove the original inequality, and this is usually much easier because $g(a,b)$ is not restrained to be close to 0. In spite of showing some examples of these transformations, we follow a more intriguing approach: we show how some interesting inequalities are created using this principle. Let's begin with some familiar identities:

$$\begin{aligned}
a^2+b^2-2ab &= (a-b)^2\\
(a+b)^2-4ab &= (a-b)^2\\
2(a^2+b^2)-(a+b)^2 &= (a-b)^2\\
a^3+b^3-ab(a+b) &= (a+b)(a-b)^2\\
a^4+b^4-ab(a^2+b^2) &= (a^2+ab+b^2)(a-b)^2
\end{aligned}$$

[1]These inequalities are also well-known by the name CAUCHY and BUCHIAKOPSKI, in that respective order.

An astute reader will notice that the first three equalities are very strict, in the sense that is very close to 0, but the fourth and the fifth one can be further reduced. We can create new inequalities from these two equalities, for example, using the assessment $a^2 + ab + b^2 \geq 3ab$, we derive something interesting.

$$\begin{array}{lrcl} & a^4 + b^4 - ab(a^2 + b^2) & \geq & 3ab(a-b)^2 \\ \Leftrightarrow & a^4 + b^4 + 6a^2b^2 & \geq & 4ab(a^2 + b^2) \end{array}$$

The last inequality is stricter and harder than $a^4 + b^4 \geq ab(a^2 + b^2)$: it consists an entity of $6a^2b^2$ on the R.H.S that is smaller than $3ab(a^2 + b^2)$ on the L.H.S. This inequality intuitively cannot be solved using CAUCHY inequality (using it makes some factor smaller), and the only way to solve it is to transform it to $(a-b)^4$, which is greater than or equal to 0.

Aside from polynomial entities, we can also have S.O.S form of fraction or root entities, such as the examples below:

$$\begin{array}{rcccl} \sqrt{2(a^2+b^2)} - (a+b) & = & \dfrac{2(a^2+b^2) - (a+b)^2}{\sqrt{2(a^2+b^2)} + a + b} & = & \dfrac{(a-b)^2}{\sqrt{2(a^2+b^2)} + a + b} \\ \dfrac{1}{a} + \dfrac{1}{b} - \dfrac{4}{a+b} & = & \dfrac{(a+b)^2 - 4ab}{ab(a+b)} & = & \dfrac{(a-b)^2}{ab(a+b)} \end{array}$$

From the first equality, using the assessment $2\sqrt{2(a^2+b^2)} \geq \sqrt{2(a^2+b^2)} + a + b \geq 2(a+b)$, we can derive the following inequality:

$$a + b + \frac{(a-b)^2}{2\sqrt{2(a^2+b^2)}} \quad \leq \quad \sqrt{2(a^2+b^2)} \quad \leq \quad a + b + \frac{(a-b)^2}{2(a+b)}$$

I guess now you can has the same feeling as me that S.O.S is the only way to untangle the above problem ◡̈. Now you can get your hand dirty by practicing S.O.S on inequalities of two variables, and being ready to observe new secrets that we have not yet tell by solving the proposed problems below.

2.1 Proposed Problems

Problem 2.1 (Tuan Anh Tran). *Let a, b be positive real numbers. Prove that:*

$$\frac{a^2}{b} + \frac{b^2}{a} + 7(a+b) \geq 8\sqrt{2(a^2+b^2)}$$

Proof. The inequality is equal to:

$$(a-b)^2\left(\frac{1}{a} + \frac{1}{b} - \frac{8}{\sqrt{2(a^2+b^2)} + a + b}\right) \geq 0$$

It is sufficient to prove the entity in the second bracket is non-negative. Indeed, we have

$$\begin{array}{ll} & \dfrac{1}{a} + \dfrac{1}{b} \geq \dfrac{4}{a+b} \geq \dfrac{8}{\sqrt{2(a^2+b^2)} + a + b} \\ \Rightarrow & \dfrac{1}{a} + \dfrac{1}{b} - \dfrac{8}{\sqrt{2(a^2+b^2)} + a + b} \geq 0 \end{array}$$

Thus, the inequality is proved. □

A challenge in solving inequality using S.O.S is that we need to derive the inequality into S.O.S form. Solving the above problem gives us some insights into this issue. In this problem, we derive the S.O.S form using two mediating equalities.

$$\begin{aligned}
\frac{a^2}{b}+\frac{b^2}{a}-a-b &= \frac{a^3+b^3-a^2b-b^2a}{ab} = \frac{(a+b)(a-b)^2}{ab} \\
8(\sqrt{2(a^2+b^2)}-a-b) &= 8\cdot\frac{2(a^2+b^2)-(a+b)^2}{\sqrt{2(a^2+b^2)}+a+b} = \frac{8(a-b)^2}{\sqrt{2(a^2+b^2)}+a+b}
\end{aligned}$$

Subtracting the second entity from the first, we obtain the S.O.S form used in the proof. So the trick here is that we subtract $\frac{a^2}{b}+\frac{b^2}{a}$ and $\sqrt{2(a^2+b^2)}$ by $a+b$, which is the arithmetic mean value (we can also use the geometric mean value ab, but in this case the arithmetic mean value is already available from the inequality). We then try to find a factor for $a+b$ so that after subtracting we get an entity that will equal to 0 when $a=b$. In that way, the square expression $(a-b)^2$ will appear.

Problem 2.2 (VMO for Secondary Student-1995)**.** *Given $x, y \neq 0$, prove that:*

$$\left(\frac{x}{y}\right)^2+\left(\frac{y}{x}\right)^2 \geq 3\left(\frac{x}{y}+\frac{y}{x}\right)-4$$

Problem 2.3 (Math & Youth Magazine)**.** *Prove that the following inequality holds for all positive real numbers a and b:*

$$\frac{a+b}{2} \geq \sqrt{ab}+\frac{(a+3b)(3a+b)(a-b)^2}{16(a+b)^3}$$

Problem 2.4 (Mathlinks.ro)**.** *Given $a, b \geq \frac{1}{2}$, prove that:*

$$\left(\frac{a^2-b^2}{2}\right)^2 \geq \sqrt{\frac{a^2+b^2}{2}}-\frac{a+b}{2}$$

3 S.O.S Form of Three Variables Inequalities

In this section, we discuss the application of S.O.S for inequalities with three or more variables. Generally, a S.O.S solution to a problem of proving $f(a_1, a_2, \ldots, a_n) \geq 0$, in which f is a function from $\mathcal{R}^n$ to $\mathcal{R}^n$ is as follows. We first transform the function $f(a_1, a_2, \ldots, a_n)$ into the form $\sum\limits_{1\leq i<j\leq n} g_{ij}(a)(a_i-a_j)^2$, which we call the S.O.S form of f. Here we use the notation $g_{ij}(a)$ to denote a function over n variables $a_1, a_2, \ldots, a_n$. We then access the value of $g_{ij}(a)$. In the simplest case, when the values of $g_{ij}(a)$ are all greater than equal to 0, the inequality is proved. Let's go through some examples with three variables inequalities.

Problem 3.1 (L. Panaitopol, V. Bandila, M. Lascu in Inegalitati Book)**.** *Prove the following inequality where $0 \leq x \leq y \leq z$:*

$$(x+y+z)(xy+yz+zx) \geq 9xyz+(y-x)(z-x)^2$$

Proof. We have:

$$\begin{array}{lrcl} & (x+y+z)(xy+yz+xz)-9xyz & = & z(x-y)^2+x(y-z)^2+y(z-x)^2 \\ \Leftrightarrow & (x+y+z)(xy+yz+xz)-9xyz-(y-x)(z-x)^2 & = & z(x-y)^2+x(y-z)^2+x(z-x)^2 \end{array}$$

Since $x, y, z \geq 0$, the L.H.S entity is greater than or equal to 0, so is the R.H.S entity, thus the inequality is proved. □

Again to transform the original inequality into S.O.S form, we use the following equalities. Observe that the equality is actually derived from a familiar inequality (≥ 0), and the entities are grouped to form S.O.S in the same way that they are grouped to prove the inequality using CAUCHY inequality.

$$\begin{array}{rl} & (x+y+z)(xy+yz+zx)-9xyz \\ = & (x^2z+y^2z-2xyz)+(xy^2+xz^2-2xyz)+(x^2y+yz^2-2xyz) \\ = & z(x-y)^2+x(y-z)^2+y(z-x)^2 \end{array}$$

3.1 Proposed Problems

Problem 3.2 (Anh Cuong Nguyen)**.** *Let $a, b, c > 0$ and $a+b+c=1$. Prove that:*

$$\frac{a^2+3b}{b+c}+\frac{b^2+3c}{c+a}+\frac{c^2+3a}{a+b} \geq 5$$

Problem 3.3 (Nam Dung Tran)**.** *Prove the following inequality for $a, b, c > 0$:*

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+a+b+c \geq 2\sqrt{3(a^2+b^2+c^2)}$$

3.2 S.O.S Theorem

The above problems deal with the simplest cases of S.O.S in which the entities $g_{ij}(a)$ are greater than or equal to 0. What we can do when these entities could be negative? The following S.O.S theorem summarizes three mostly used techniques that one can use to tackle this problem.

Theorem 3.1 (S.O.S Theorem for Three Variables Inequalities)**.** *Let a, b, c be real numbers and A, B, C be real entities that satisfy one of these following conditions:*

1. *$A+B \geq 0, B+C \geq 0$ and $C+A \geq 0$. If the value of c is in between a and b then $C \geq 0$. The latter condition can be alternatively applied for a and b in a similar manner.*
2. *$A+B+C > 0$ and $AB+AC+BC \geq 0$.*
3. *If $C = min\{A, B, C\}$ then $A+2C \geq 0$ and $B+2C \geq 0$. This condition can be alternatively applied for A and B in a similar manner.*

Then we can conclude that $A(b-c)^2+B(c-a)^2+C(a-b)^2 \geq 0$.

Proof.

1. Assume that c is in the middle of a and b (the other two cases can be considered similarly), denote $b-c$ as x and $c-a$ as y. The inequality is equal to:

$$\begin{array}{rll} & Ax^2+By^2+C(x+y)^2 & \geq 0 \\ \Leftrightarrow & (A+C)x^2+(B+C)y^2+2Cxy & \geq 0 \end{array}$$

   Notice that $(A+C), (B+C)$ and Cxy are all greater than or equal to 0, thus the inequality holds.

2. If $ABC = 0$, W.L.O.G assume that $C = 0$. We then have $A+B \geq 0$ and $AB \geq 0$, which lead to $A \geq 0$ and $B \geq 0$, thus the inequality holds. Otherwise, we can also prove that at least two in three entities A, B and C are positive.

   Indeed assume that there is at least one among them negative, and W.L.O.G assume that is C. When then first have $A+B > 0$. This in turn leads to $C(A+B) < 0$. But because $C(A+B)+AB \geq 0$, AB must be positive. The fact that $A+B > 0$ and $AB > 0$ show that A and B are positive.

   We then have the following inequality, by applying BUCHIAKOPSKI inequality:

$$\begin{array}{rllll} & \left(\frac{1}{A}+\frac{1}{B}\right)\left[A(b-c)^2+B(c-a)^2\right] & \geq (b-c+c-a)^2 & = & (a-b)^2 \\ \Leftrightarrow & A(b-c)^2+B(c-a)^2 & \geq \left(\frac{AB}{A+B}\right)(a-b)^2 & & \\ \Leftrightarrow & A(b-c)^2+B(c-a)^2+C(a-b)^2 & \geq \left(\frac{AB}{A+B}+C\right)(a-b)^2 & = & \left(\frac{AB+BC+CA}{A+B}\right)(A-B)^2 \end{array}$$

   The L.H.S entity is greater or equal than 0 by the fact that $AB+BC+CA \geq 0$ and $A+B > 0$, and so is the R.H.S entity.

3. Assume that $C = min\{A, B, C\}$. If $C \geq 0$, then all A, B and C are greater than or equal to 0, so the inequality trivially holds. Assume that $C \leq 0$, we have:

$$\begin{array}{rll} & C(a-b)^2 = C(b-c+c-a)^2 & \geq 2C(b-c)^2+2C(c-a)^2 \\ \Leftrightarrow & A(b-c)^2+B(c-a)^2+C(a-b)^2 & \geq (2C+A)(b-c)^2+(2C+B)(c-a)^2 \end{array}$$

   The L.H.S entity is greater or equal than 0 by the assumption that $2C+A \geq 0$ and $2C+B > 0$, and so is the R.H.S entity.

□

Let try out the theorem in some of following examples.

Problem 3.4 (Anh Cuong Nguyen)**.** *Let a, b and c be real numbers such that $a+b+c=1$. Prove that:*

$$4(a^3+b^3+c^3-3abc) \geq 3(a^2+b^2-2ab)$$

Proof. The inequality is equal to:

$$\begin{array}{rll} & 2(a+b+c)\left[(a-b)^2+(b-c)^2+(c-a)^2\right] & \geq 3(a-b)^2 \\ \Leftrightarrow & 2(b-c)^2+2(c-a)^2-(a-b)^2 & \geq 0 \end{array}$$

Using the notation given in the theorem, we have in this S.O.S form, $A=2, B=2$ and $C=-1$. Notice that $A+B+C(=3) > 0$ and $AB+BC+CA(=0) \geq 0$, therefore applying the second condition of the theorem, the inequality holds. □

Notice that the equality of the above inequality can occur, not only when $a = b = c = \frac{1}{3}$, but also in general $a + b = 2c$. An astute reader may notice that the satisfaction of the second condition of the S.O.S theorem can introduce one more equality point for the inequality ◡̈.

Problem 3.5 (Anh Cuong Nguyen). *Let a, b, c be non-negative real numbers. Prove that:*

$$a^3 + b^3 + c^3 + 3abc \geq ab\sqrt{2(a^2+b^2)} + bc\sqrt{2(b^2+c^2)} + ca\sqrt{2(c^2+a^2)}$$

Proof. The first observation one may have is that this inequality is an improvement of SCHUR inequality for three variables, which is:

$$a^3 + b^3 + c^3 + 3abc \geq ab(a+b) + bc(b+c) + ca(c+a)$$ We untangle this inequality using S.O.S theorem. First, W.L.O.G, we can assume that $a \geq b \geq c$. The inequality can be re-written in S.O.S form as:

$$\begin{aligned} &\left(a+b-c-\frac{2ab}{\sqrt{2(a^2+b^2)}+a+b}\right)(a-b)^2 \\ +\ &\left(-a+b+c-\frac{2bc}{\sqrt{2(b^2+c^2)}+b+c}\right)(b-c)^2 + \left(a-b+c-\frac{2ca}{\sqrt{2(c^2+a^2)}+c+a}\right)(c-a)^2 \geq 0 \end{aligned}$$

Using the first condition of the S.O.S theorem, it is sufficient to prove that:

$$B = a - b + c - \frac{2ac}{\sqrt{2(a^2+c^2)}+a+c} \geq 0$$

This is indeed true because $\dfrac{2ac}{\sqrt{2(a^2+c^2)}+a+c} \geq \dfrac{2c}{\sqrt{2}+1}$, which leads to:

$$B \geq (a-b) + c\left(1 - \frac{2}{\sqrt{2}+1}\right) \geq 0$$

We also need to prove that $A + B$, $B + C$ and $C + A$ are non-negative. The following proof shows that $A + B \geq 0$. Other cases can be proved similarly.

$$\begin{aligned} A + B &= 2c - \frac{2ac}{\sqrt{2(a^2+c^2)}+a+c} - \frac{2bc}{\sqrt{2(b^2+c^2)}+b+c} \\ &\geq 2c\left(1 - \frac{a}{2(a+c)} - \frac{b}{2(b+c)}\right) \\ &= c\left(\frac{c}{a+c} + \frac{c}{b+c}\right) \geq 0 \end{aligned}$$

The inequality therefore holds according to the first condition of S.O.S theorem. □

Problem 3.6 (Vasile Cirtoaje). *Let a, b, c be three side lenghts of a triangle. Prove that:*

$$3\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \geq 2\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right) + 3$$

Proof. The inequality can be re-written to:

$$(5a-5b+3c)(a-b)^2+(5b-5c+3a)(b-c)^2+(5c-5a+3b)(c-a)^2 \quad \geq \quad 0$$

Denote $a+b$, $b+c$ and $c+a$ as x, y and z in that respective order. Notice that if $x \geq y \geq z$, it would be not too difficult to prove that:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \quad \geq \quad \frac{b}{a}+\frac{c}{b}+\frac{a}{c}$$

Therefore, the inequality trivilly holds.

Assume that $z \geq y \geq x$. The inequality can be re-written in term of x, y, z as follows:

$$(4x-z)(x-z)^2+(4z-y)(y-z)^2+(4y-x)(x-y)^2 \quad \geq \quad 0$$ Observe that $B = 4x - z$ is the minimum among A, B and C (otherwise all three factors A, B and C are non-negative and the inequality holds trivially. Using the third condition of S.O.S theorem, we need to prove that:

$$\begin{aligned} A+2B &= 7x+4y-2z &\geq 0 \\ \& \quad C+2B &= 8x+2z-y &\geq 0 \end{aligned}$$

Observe that $C_+2B \geq 0$ because $z \geq 0$. If $A+2B \geq 0$ holds also, the inequality is proved according to the third condition of S.O.S theorem. Otherwise we have $z \geq 2y$, which leads to $z-y \geq y-x \geq 0$. Therefore:

$$\begin{aligned} (4x-z)(x-z)^2+(4z-y)(y-z)^2+(4y-x)(x-y)^2 &\geq (A+2B)(x-y)^2+(C+2B)(y-z)^2 \\ &\geq (A+C+4B)(x-y)^2 \\ &= (15x+3y)(x-y)^2 \end{aligned}$$

Thus the inequality also holds. □

The last example we want to show you is the well-known hard Iran Math Olympiad 1996 problem. The official solution requires a total expansion of the inequality into polynomial form, and then applies SCHUR and CAUCHY inequalities in proper group of polynomial entities. We show another solution for this problem, using S.O.S theorem.

Problem 3.7. *Let a, b, c be three positive numbers. Prove that:*

$$(ab+bc+ca)\left[\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right] \quad \geq \quad \frac{9}{4}$$

Proof. Re-write the inequality into the following S.O.S form.

$$\begin{aligned} &\left[\frac{2}{(a+c)(b+c)}-\frac{1}{(a+b)^2}\right](a-b)^2+\left[\frac{2}{(a+b)(a+c)}-\frac{1}{(b+c)^2}\right](b-c)^2 \\ + &\left[\frac{2}{(a+b)(b+c)}-\frac{1}{(a+c)^2}\right](c-a)^2 \qquad \geq \quad 0 \end{aligned}$$

$$\begin{aligned} \Leftrightarrow &\left[\frac{2}{c^2(a+c)(b+c)}-\frac{1}{c^2(a+b)^2}\right](ac-bc)^2+\left[\frac{2}{a^2(a+b)(a+c)}-\frac{1}{a^2(b+c)^2}\right](ab-ac)^2 \\ + &\left[\frac{2}{b^2(a+b)(b+c)}-\frac{1}{b^2(a+c)^2}\right](bc-ba)^2 \qquad \geq \quad 0 \end{aligned}$$

Using the notation of S.O.S theorem, we denote $A = \frac{2}{a^2(a+b)(a+c)} - \frac{1}{a^2(b+c)^2}$, $B = \frac{2}{b^2(b+a)(b+c)} - \frac{1}{b^2(a+c)^2}$, and $C = \frac{2}{c^2(c+b)(c+a)} - \frac{1}{c^2(a+b)^2}$.

W.L.O.G, assume that $a \geq b \geq c$ then it is not too difficult to prove that $B \geq 0$ (basically because $2(a+c)^2 > (a+b)(c+b)$), so according to the first condition of S.O.S theorem, it is sufficient to prove that: $A+B, B+C$ and $C+A$ are non-negative. We will prove here that $A+B \geq 0$, the other two conditions can be proved similarly.

$$\begin{aligned}
A+B &= \frac{2}{a^2(b+a)(b+c)} + \frac{2}{b^2(b+a)(b+c)} - \frac{1}{a^2(b+c)^2} - \frac{1}{b^2(c+a)^2} \\
&> \frac{2}{a+b}\left[\frac{1}{a^2(a+c)} + \frac{1}{b^2(b+c)}\right] - \frac{1}{a^2b(b+c)} - \frac{1}{b^2a(a+c)} \\
&= \frac{1}{a(a+c)}\left[\frac{2}{a(a+b)} - \frac{1}{b^2}\right] + \frac{1}{b(b+c)}\left[\frac{2}{b(a+b)} - \frac{1}{a^2}\right] \\
&= \frac{(b-a)(2b+a)}{a^2b^2(a+c)(a+b)} + \frac{(a-b)(2a+b)}{a^2b^2(b+c)(a+b)} \\
&= \frac{(2a+2b+c)(a-b)^2}{a^2b^2(a+b)(b+c)(c+a)} \geq 0
\end{aligned}$$

The inequality therefore holds according to the first condition of the S.O.S theorem. □

What notable here is our transformation of the original inequality into its S.O.S form. From the standard form $A(b-c)^2 + B(c-a)^2 + C(a-b)^2 \geq 0$, one can further transform it into:

$$\frac{A}{a^2}(ab-ac)^2 + \frac{B}{b^2}(bc-ba)^2 + \frac{C}{c^2}(ca-cb)^2 \geq 0$$

Or

$$Ab^2c^2\left(\frac{1}{b} - \frac{1}{c}\right)^2 + Ba^2c^2\left(\frac{1}{c} - \frac{1}{a}\right)^2 + Ca^2b^2\left(\frac{1}{a} - \frac{1}{b}\right)^2 \geq 0$$

New conditions for the inequality to hold can be derived accordingly from the new form. For example, conditions for the former transformation are as follows:

1. $Ab^2 + Ba^2 \geq 0, Bc^2 + Cb^2 \geq 0$ and $Ca^2 + Ac^2 \geq 0$. If the value of c is in the middle of a and b then $C \geq 0$. The latter condition can also be alternatively applied for a and b.
2. $Ab^2c^2 + Ba^2c^2 + Ca^2b^2 > 0$ and $ABc^2 + ACb^2 + BCa^2 \geq 0$.
3. If $\frac{C}{c^2} = min\{\frac{A}{a^2}, \frac{B}{b^2}, \frac{C}{c^2}\}$ then $Ac^2 + 2Ca^2 \geq 0$ and $Bc^2 + 2Cb^2 \geq 0$. This condition can also be alternatively applied for A and B.

We conclude this section with some practicing problems: all of them can be proved using the S.O.S theorem and the transformations introduced.

3.3 Proposed Problems

Problem 3.8 (Anh Cuong Nguyen). *Let a, b, c be three positive real numbers. Prove that:*

$$a^3 + b^3 + c^3 + 6(ab^2 + bc^2 + ca^2) \geq 3(a^2b + b^2c + c^2a) + 12abc$$

Problem 3.9 (Anh Cuong Nguyen). *Let* a, b, c *be positive number such that* $\sqrt{ab}, \sqrt{ac}$ *and* $\sqrt{bc}$ *are lengths of a triangle. Prove that:*

$$a^2b(a-b) + b^2c(b-c) + c^2a(c-a) \geq 0$$

Problem 3.10 (Vasile Cirtoaje). *If* a, b, c *are positive numbers such that* $a+b+c=3$*, prove that:*

$$\frac{a}{b^2+2} + \frac{b}{c^2+2} + \frac{c}{a^2+2} \geq 1$$

Problem 3.11 (Anh Cuong Nguyen). *Prove the following inequality given that* $a, b, c > 0$*:*

$$\frac{abc}{a^3+b^3+c^3} + \frac{2}{3} \geq \frac{ab+bc+ca}{a^2+b^2+c^2}$$

Problem 3.12 (Anh Cuong Nguyen). *Transform the following expression into* S.O.S *form and prove it given that* $x, y, z > 0$*:*

$$\sqrt{\frac{x^4+y^4+z^4}{x^2y^2+y^2z^2+z^2x^2}} + \sqrt{\frac{2(xy+yz+zx)}{x^2+y^2+z^2}} \geq 1+\sqrt{2}$$

4 Conclusion

We introduce S.O.S, a novel technique for proving inequalities systematically and elegantly. More importantly, we show how a birth of a technique has been created, from solving concrete problems and then accumulating observations and ideas. The technique is just a grain of sand in our vast desert of knowledge. There are more and more ideas hidden inside every problems that are still waiting for you to wake them up. Hope you have enjoyed the reading, and this small article has given you something, anything that you find useful and can boost you forward ⌣̈.