Adaptive Differentially Private Histogram of Low-Dimensional Data

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Background: Differential Privacy

A mechanism $A$ achieves (Bounded) $\varepsilon$-Differential Privacy, if

$$e^{-\varepsilon} \leq \frac{\Pr[A(D) = a]}{\Pr[A(D') = a]} \leq e^\varepsilon$$

for any published $a$ and any pair of “neighbouring” datasets $D$ and $D'$. 
“Bounded” diff. privacy

D and D’ are neighbours

iff

D’ can be obtained from D by replacing one element.
Background: Sensitivity

The sensitivity of $f: \mathcal{D} \rightarrow \mathbb{R}^n$, denoted as $\Delta f$, is defined as:

$$\Delta f = \max_{D, D'} |f(D) - f(D')|_1$$

where max is taken over all neighbouring $D, D'$. 

sensitivity $\Delta f$ is a bound on the difference.
Background: Sensitivity $\rightarrow$ diff. priv. [Dwork06]

If sensitivity of a function $f$ is $\Delta f$ then the mechanism $A$

$$A(D) = f(D) + \text{LAP}(\Delta f / \epsilon)$$

achieves $\epsilon$-differential privacy.
Problem: illustrating examples

• We want to publish the “distribution” of a dataset $D$ in a differentially private manner.

  – e.g. incomes of a group of taxpayers,
    \[ D = \{ \$10031, \$8931, \$3001, \$21530, \ldots, \$32320 \} \]

  – e.g. Locations of individuals

![Diagram of Low-Dimensional Data]
Existing approach: Equi-width Histogram

The actual histogram with 30 bins.

Sensitivity is 2
Existing approach: Equi-width Histogram

Adding Laplace noise to the counts.

Adding Laplace noise of LAP(2/ε) can achieve $\varepsilon$-differential privacy$^{[Dwork06]}$
Existing approach: Equi-width Histogram

The published noisy histogram.
Problem with Equi-Width Histogram

Small bin-width:
Incur too much noise.

Large bin-width:
Lost detail information.
Enhancements and variations


• Exploit dependencies in the published data [Li10, Barak07, Hay10].

• Construct varying bin-width histograms from previously released data [Machanavajjhala08], synthetic data [Xiao11], and from an equi-width histogram [Xu12].
Instead of adding noises to the frequency counts, can we publish the data directly?
Our Approach: main idea

• Sort the data; add noise directly to the data; and publish the noisy data.

\[
D \\
\downarrow \text{sorting} \\
S(D) = \langle x_1, x_2, x_3, \ldots, x_m \rangle \\
\downarrow \text{add Laplace noise} \\
S(D)' = \langle x_1 + n_1, x_2 + n_2, x_3 + n_3, \ldots, x_m + n_m \rangle
\]
Our Approach: main idea

• Sort the data; add noise directly to the data; and publish the noisy data.

\[ D \]

*Will the published data too noisy?*

\[
S(D) = \langle x_1, x_2, x_3, \ldots, x_m \rangle
\]

\[
S(D)' = \langle x_1 + n_1, x_2 + n_2, x_3 + n_3, \ldots, x_m + n_m \rangle
\]
Our Approach: main idea

• Sort the data; add noise directly to the data; and publish the noisy data.

$$D$$

**Will the published data too noisy?**

$$S(D) = \langle x_1, x_2, x_3, \ldots, x_m \rangle$$

**How to extend to higher dimension?**

$$S(D)' = \langle x_1 + n_1, x_2 + n_2, x_3 + n_3, \ldots, x_m + n_m \rangle$$
Observations & Techniques

1. Show that the sensitivity of “sorting” is not too large.

2. Exploit redundancy using Isotonic regression.

3. Grouping to tradeoff generalization errors with the level of Laplace noise.

4. Extension to higher dimension through location preservation mapping.
1. Sensitivity

For two neighbouring $D$ and $D' \subseteq [0,1]$

$\text{Sort}(D) = \langle x_1, x_2, x_3, x_4, \ldots, x_{m-2}, x_{m-1}, x_m \rangle$

$\text{Sort}(D') = \langle x_1, x_3, x_4, \ldots, x_{m-2}, x_{m-1}, x_m, 1 \rangle$
1. Sensitivity

For two neighbouring $D$ and $D' \subseteq [0,1]$

\[
\text{Sort}(D) = \langle x_1, x_2, x_3, x_4, \ldots, x_{m-2}, x_{m-1}, x_m \rangle
\]

\[
\text{Sort}(D') = \langle x_1, x_3, x_4, \ldots, x_{m-2}, x_{m-1}, x_m, 1 \rangle
\]

\[| \text{sort } (D) - \text{sort } (D') |_1 \]
1. Sensitivity

For two neighbouring $D$ and $D' \subseteq [0,1]$

\[
\text{Sort}(D) = \langle x_1 \, x_2 \, x_3 \, x_4 \, \ldots \, x_{m-2} \, x_{m-1} \, x_m \rangle
\]

\[
\text{Sort}(D') = \langle x_1 \, x_3 \, x_4 \, \ldots \, x_{m-2} \, x_{m-1} \, x_m \, 1 \rangle
\]

\[
|\text{sort}(D) - \text{sort}(D')|_1 \leq 1
\]
2. Isotonic regression

Note that the sorted data are constrained: the elements are increasing.

Isotonic regression: Given a sequence

\[ Y = \langle y_1, y_2, y_3, \ldots, y_m \rangle \]

find an non-decreasing sequence

\[ X = \langle x_1, x_2, x_3, \ldots, x_m \rangle \]

minimizing the distance of \( X \) from \( Y \).
3. Grouping

Group consecutive elements and publish its noisy sum.

\[ \text{Sort}(D) = \left\langle \begin{array}{cccccccc}
  x_1 & x_2 & x_3 & x_4 & \ldots & x_{m-2} & x_{m-1} & x_m \\
\end{array} \right\rangle \]

\[ x_1 + x_2 + x_3 \quad x_4 + x_5 + x_6 \quad \ldots \quad x_{m-2} + x_{m-1} + x_m \]
3. Grouping

Group consecutive elements and publish its noisy sum.

\[ \text{Sort}(D) = \langle x_1, x_2, x_3, x_4, \ldots, x_{m-2}, x_{m-1}, x_m \rangle \]

\[ x_1 + x_2 + x_3 \quad x_4 + x_5 + x_6 \quad \ldots \quad x_{m-2} + x_{m-1} + x_m \]

Grouping does not affect sensitivity.
3. Grouping

Group consecutive elements and publish its noisy sum.

\[ \text{Sort}(D) = \langle x_1, x_2, x_3, x_4, \ldots, x_{m-2}, x_{m-1}, x_m \rangle \]

\[ \begin{align*}
&x_1 + x_2 + x_3 + \text{noise} \\
&x_4 + x_5 + x_6 + \text{noise} \\
&\vdots \\
&x_{m-2} + x_{m-1} + x_m + \text{noise}
\end{align*} \]

Grouping does not affect sensitivity.
Illustration

The Grouped Sorted data

Adaptive Differentially Private Histogram of Low-Dimensional Data
Illustration

With Laplace Noise
Illustration

Isotonic regression.
Grouping: what should be the appropriate group size?

We give a model to estimate the expected error based on the (1) group size $k$, (2) size of dataset $n$ and (3) privacy requirement $\epsilon$.

From the model, we can estimate the optimal group size $k$, given $n$ and $\epsilon$. 
Expected_error (ε, k, n) ≈

Generalization_error (n, k) +
Laplace_noise (ε, n, k)
Expected_error \( (\varepsilon, k, n) \) \approx

\[ \text{Generalization_error} \ (n, k) + \] \n
\[ k^{-1} \text{ Laplace\_noise\_without\_grouping} \ (\varepsilon, n k^{-1}) \]
Expected_error (ε, k, n) ≈

Generalization_error (n, k) + k^{-1} Laplace_noise_without_grouping (ε, n k^{-1})
Accuracy of Error Model

Kaluza's data: [Kaluza10]  Twitter data: [Twitterdata10]
4. Extension to Higher Dimension

- Consider location preserving mapping
  \[ T: [0,1] \times [0,1] \rightarrow [0,1] \]
  s.t.,
  if \( T(x) \) and \( T(y) \) are “close-by” in \([0,1]\)
  then \( x, y \) are “close-by” in \([0,1] \times [0,1]\)
Extension to Higher Dimension

- Example of such mapping: Hilbert space filling curve.

2D data points  
Location preserving mapping  
Sorted 1D data points
Putting all together: Proposed mechanisms

Given the dataset $D$, privacy requirement $\epsilon$, the publisher performs:

1. Determines the group size $k$ from $n=|D|$, and $\epsilon$.
2. Maps $D$ to $[0,1]$. Let the mapped points be $T(D)$.
3. Sorts $T(D)$.
4. Groups $k$ consecutive elements.
5. Adds noise to the sum in each group. Publishes the noisy sums.

Given the published data, a user performs:

1. Isotonic regression.
2. Inverse of the location preserving mapping.
3. Subsequent operations, like query, visualization, & data mining.
Evaluation: Datasets

• Profile of Twitter users. [Twitterdata10]

Locations of 180,000 profiles in North America.

• The distance of the locations to New York City is taken as the 1D data.
Adaptive Resolution

- A visualization of our method and equi-width histogram
Evaluation: Range Query

- We repeat the experiment 1,000 times for each size of the range $q$. We compare our algorithm with equi-width histogram and wavelet-based method [Xiao10].
Range Query: 2D domain
Range Query: 1D domain
Evaluation: Median-Finding

- We compare our algorithm with the smooth-sensitivity approach [Nissim07].
Evaluation: Median-finding

![Graph showing error rates for different values of $\varepsilon$. The graph compares 'Our method' with 'Smooth sensitivity based method'.]
Discussion: Complementary

- Alternative ``direction’’ of the Laplace noise
Conclusion

• We proposed an approach that publishes the data directly.
  – Simple.
  – The main parameter (group size) can be determined without the dataset D. In contrast, optimal parameters of many existing mechanisms heavily rely on the dataset.
  – Leads to adaptive histograms. Achieve high utility.
  – Complementary to the frequency-counts methods and potentially can be combined for higher utility.

• We proposed using location preservation mapping for extension to low-dimensional data (for e.g. 2D and 3D).
Reference

• [Dwork06]: C. Dwork. Differential privacy. Automata, languages and programming, page 1, 2006.