Introduction

A function from a non-empty set to another is a relation from the set to another satisfying two properties. We will discuss

1. When does a relation become a function?
2. Terminology related to functions: images, pre-images, domains, co-domains, ranges, etc.
3. One-to-one functions, onto functions, one-to-one correspondences.
4. Inverse of functions.
5. Function composition.

Functions

A function f from a non-empty set A to a set B is a relation from A to B satisfying the following two properties:

1. $\forall x \in A$, $\exists y \in B$ such that $(x, y) \in f$.

2. $\forall (x, y), (x, y') \in f$, $y = y'$.

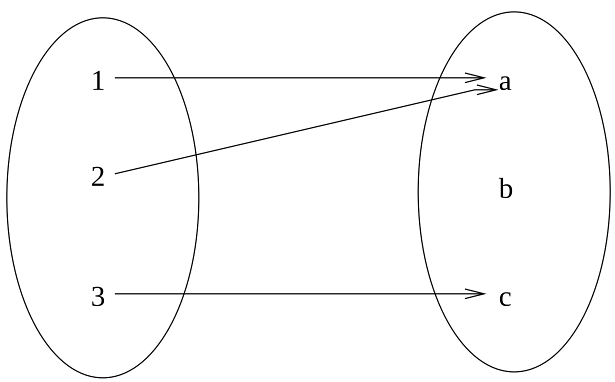
The first property says every $x \in A$ is related to at least one $y \in B$. The second property says each $x \in A$ is related to at most one $y \in B$.

That is, a relation from A to B is a function from A to B if and only if every $x \in A$ is related to exactly one $y \in B$.

The Arrow Diagram of Functions

The arrow diagram of a function from A to B has the characteristic that there is *exactly one* arrow shooting out from *every element* of A.

However, a element of B can be hit by no arrows, one arrow, or many arrows. Example:



Example: Functions

- Let $A = \{1, 2, 3\}$, $B = \{7, 8, 9, 10\}$.

- $f = \{(1, 10), (2, 8)\} \subseteq A \times B$ is not a function from A to B: $3 \in A$ is not related to any element of B. Relation f fails to be a function because $3 \in A$ is related to no elements in B.

- $g = \{(1, 8), (2, 9), (3, 9), (3, 10)\} \subseteq A \times B$ is not a function from A to B: $(3, 9), (3, 10) \in g$ but $9 \neq 10$. Relation g fails to be a function because $3 \in A$ is related to two (distinct) elements $9, 10 \in B$.

- $h = \{(1, 9), (2, 10), (3, 9)\} \subseteq A \times B$ is a function from A to B. Relation h is a function because each element of A is related to exactly one element in B.

Domains and Co-domains

We write $f : A \rightarrow B$ to mean f is a function from set A to set B. Set A is called the **domain** of f. Set B is called the **co-domain** of f.

The Value, Image of an Element under a Function

Let $R \subseteq A \times B$. For each $x \in A$, define

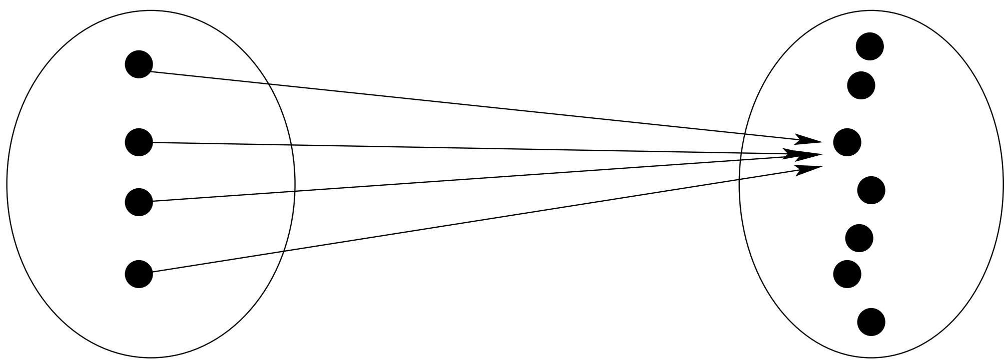
$$R(\{x\}) = \{y \in B \mid (x, y) \in R\}.$$

Note that R is a function if and only if for all $x \in A$, $|R(x)| = 1$. Consequently, when R is a function and $(x, y) \in R$, we simply write $R(x) = y$ instead of $R(\{x\}) = \{y\}$.

Let $f : A \to B$. Let $(x, y) \in f$. (Since f is a function, for each $x \in A$, such a y exists and is unique.) We say "f sends x to y" and write $y = f(x)$. We call "$f(x)$" as "f **of** x" or, "**the value of** f **at** x", or "**the image of** x **under** f".

Example: Constant Functions

Let $f : A \rightarrow B$. If for all $x, y \in A$, $f(x) = f(y)$, then f is called a constant function.



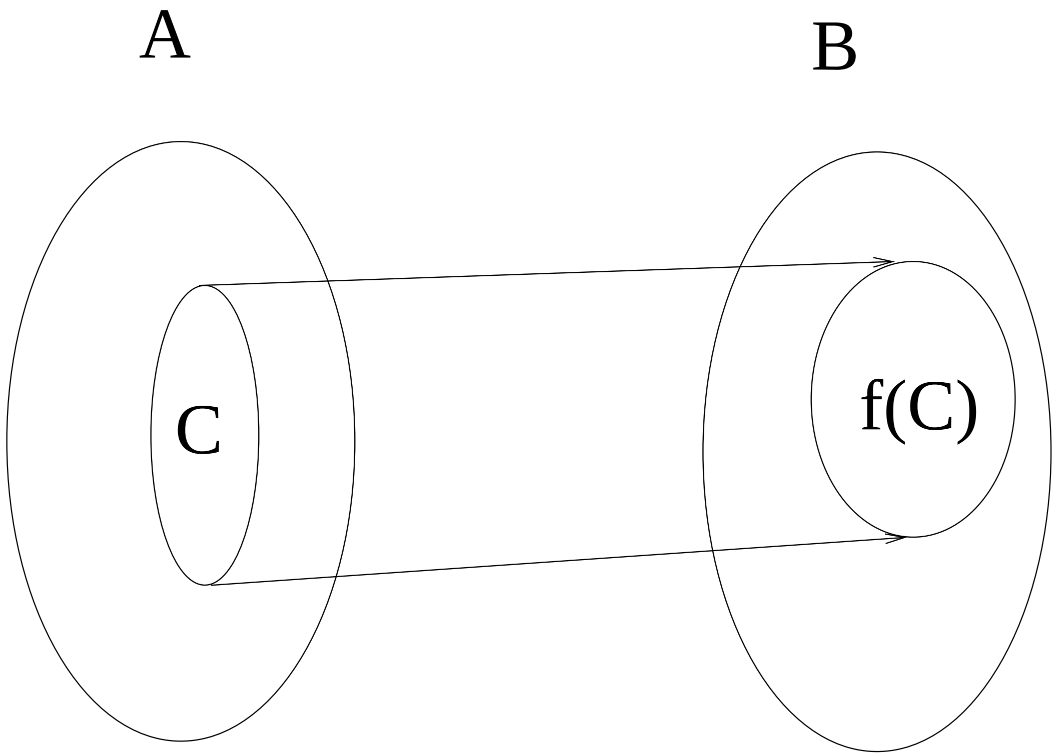
Related Subsets

For any relation $R \subseteq A \times B$ and any subset $C \subseteq A$, we define

$$R(C) = \{y \in B \mid \exists x \in C \text{ such that } (x, y) \in R\}.$$

That is, $R(C) \subseteq B$ is a set consisting of elements of B that are related to some elements of $C \subseteq A$.

If $C = \{c\}$, we may write $R(c)$ instead of $R(\{c\})$.


A
B
C
f(C)

Related Subsets

Since $R^{-1} \subseteq B \times A$ is also a relation (from B to A), similarly, for any $D \subseteq B$, we have

$$\begin{aligned} R^{-1}(D) &= \{x \in A \mid \exists y \in D \text{ such that } (y,x) \in R^{-1}\} \\ &= \{x \in A \mid \exists y \in D \text{ such that } (x,y) \in R\}. \end{aligned}$$

Examples: Related Subsets

Consider the greater-or-equal-than relation GE on $A = \{-1, 0, 1\}$:

$$GE = \{(-1,-1), (0,-1), (0,0), (1,-1), (1,0), (1,1)\}.$$

We have

$$\begin{aligned} GE(\{\}) &= \{\} \\ GE(\{-1\}) &= \{-1\} \\ GE(\{0\}) &= \{-1, 0\} \\ GE(\{1\}) &= \{-1, 0, 1\} \\ GE(\{-1, 0\}) &= \{-1, 0\} \end{aligned}$$

$$\begin{aligned} GE(\{0,1\}) &= \{-1,0,1\} \\ GE(\{-1,1\}) &= \{-1,0,1\} \\ GE(\{-1,0,1\}) &= \{-1,0,1\} \end{aligned}$$

The Range of a Function is the Image of the Domain under the Function

Let $f : A \to B$. The set $f(A) \subseteq B$ is call the **range of** f, or the **image of** A **under** f. Symbolically,

$$\begin{aligned} f(A) &= \{y \in B \mid \exists x \in A, (x,y) \in f\} \\ &= \{y \in B \mid \exists x \in A, f(x) = y\} \\ &= \{f(x) \mid x \in A\}. \end{aligned}$$

Pre-image, Inverse Image of an Element in the Co-domain

Let $f : A \to B$. If $y = f(x)$, x is called **a preimage of** y, or **an inverse image of** y. The set of preimages of y is called **the inverse image of** y. Symbolically,

$$f^{-1}(y) = \{x \in A \mid f(x) = y\}.$$

Note that if $f : A \to B$, then $f \subseteq A \times B$. So $f^{-1} \subseteq B \times A$ is a relation and $f^{-1}(y) = f^{-1}(\{y\})$ has already been defined previously. The previous definition is consistent with the present definition.

Example: Pre-image, Inverse Image of an Element in the Co-domain

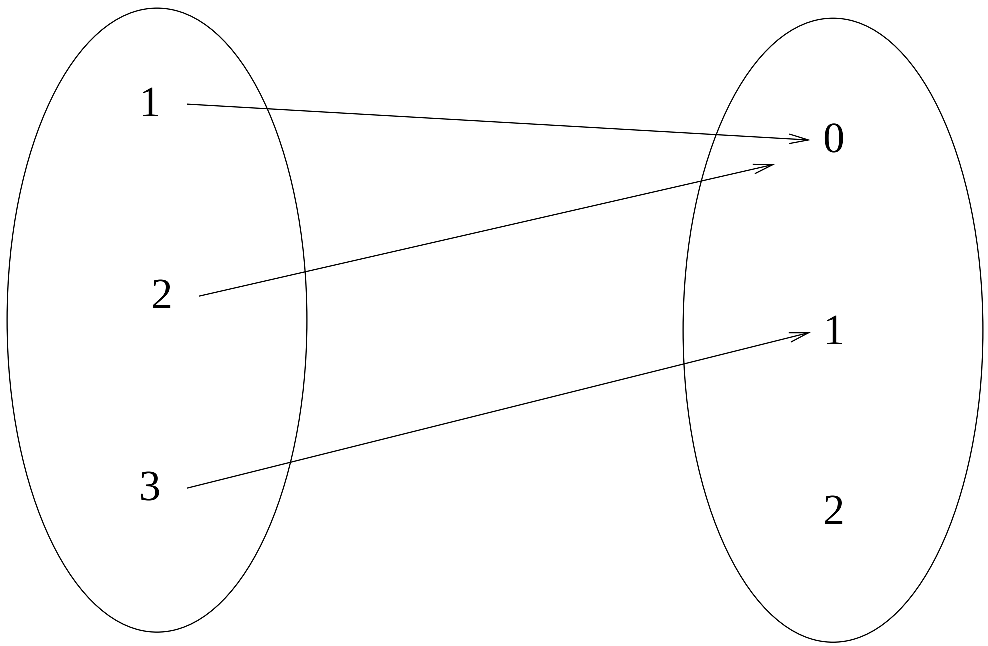
Let $f : \{1, 2, 3\} \to \{0, 1, 2\}$ be

$$f = \{(1, 0), (2, 0), (3, 1)\}.$$

We have

$$\begin{aligned} f^{-1}(0) &= \{1, 2\} \\ f^{-1}(1) &= \{3\} \\ f^{-1}(2) &= \{\} \end{aligned}$$

Example: Continued



Equality of Functions

Let $f, g : A \rightarrow B$, then f and g are equal if and only if $f = g$ as subsets of $A \times B$.

Functions Should be Well-Defined

Functions should be well-defined. This is a concern when defining a function on a domain in which an element can have multiple representations.

Let $f : \mathbf{Q} \to \mathbf{R}$ be defined as $f\left(\frac{m}{n}\right) = m$. The function is not well defined.

Write $1 = \frac{1}{1}$. Then

$$f(1) = f\left(\frac{1}{1}\right) = 1.$$

But $1 = \frac{2}{2}$ and

$$f(1) = f\left(\frac{2}{2}\right) = 2.$$

Thus the value of $f(1)$ is not properly defined and f is not well-defined.

One-to-One (1-1) Functions

Let $f : A \to B$. The function f is **one-to-one**, or **1-1**, or **injective**, if and only if for all x, $y \in A$, if $f(x) = f(y)$ then $x = y$.
Alternatively, $f : A \to B$ is **one-to-one** if and only if for all $(x, y), (x', y) \in f$, $x = x'$.

The arrow diagram of a 1-1 function has the characteristic that two different arrows cannot hit the same element.

Example: 1-1 Functions

Let $A = \{1, 2, 3\}$.

$f = \{(1, 7), (2, 3), (3, 7)\} \subseteq A \times \mathbf{Z}$ is a function but is not 1-1:

$$f(1) = f(3) = 7,\ 1 \neq 3.$$

Example: 1-1 Functions

Let $f : \mathbf{R} \to \mathbf{R}$ be given as $f(x) = 3x + 2$. Prove that f is 1-1.

Let $f(x) = f(y)$. Then

$$3x + 2 = 3y + 2$$

and consequently

$$x = y.$$

That is, f is 1-1.

Example: Not 1-1 Functions

Let $f : \mathbf{R} \to \mathbf{R}$ be given as $f(x) = x^2$. Prove that f is not 1-1.

The negation of

$$\forall x, y \in \mathbf{R}, (f(x) = f(y)) \to (x = y)$$

is

$$\exists x, y \in \mathbf{R}, (f(x) = f(y)) \wedge (x \neq y).$$

The negation is true, consider any $x \neq 0$,

$$f(x) = x^2 = (-x)^2 = f(-x) \text{ but } x \neq -x.$$

Thus f is not 1-1.

Onto Functions

Let $f : A \rightarrow B$. The function f is **onto**, or **surjective**, if and only if for any $y \in B$, there is some $x \in A$ such that $f(x) = y$. Symbolically, $f : A \rightarrow B$ is onto if and only if

$$\forall y \in B, \exists x \in A \text{ such that } f(x) = y.$$

The arrow diagram of an onto function has the characteristic that every element in the co-domain is hit by an arrow.

Example: Onto Functions

Show that $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = 2x + 3$ is an onto function. For any $y \in \mathbf{R}$, let $f(x) = y$:

$$2x + 3 = y.$$

Solving for x:

$$x = \frac{y-3}{2}.$$

Since $\frac{y-3}{2} \in \mathbf{R}$ and

$$f\left(\frac{y-3}{2}\right) = y,$$

the function f is onto.

Example: Not Onto Functions

Show that $f : \mathbf{Z} \to \mathbf{R}$ given by $f(x) = 2x + 3$ is not an onto function.

Let $y \in \mathbf{R}$ be non-integer. For any $x \in \mathbf{Z}$, $f(x) = 2x + 3$ is an integer. Thus for all $x \in \mathbf{Z}$, $f(x) \neq y$. That is, f is not onto.

One-to-One Correspondences

Let $f : A \rightarrow B$. The function f is a **1-1 correspondence**, or a **bijection**, or **bijective**, if and only if f is 1-1 and onto.

Example: One-to-One Correspondences

- Let $f : \mathbf{R} \to \mathbf{R}$ be given by $f(x) = 2x + 3$.
- It is easy to show that
  1. $\forall y \in \mathbf{R}$, $\frac{y-3}{2} \in \mathbf{R}$ and $f\left(\frac{y-3}{2}\right) = y$.
  2. $\forall x, y \in \mathbf{R}$, if $f(x) = f(y)$ then $x = y$.
- That is, f is bijective.

Example: One-to-One Correspondences

- Let $A = \{1, 2, \ldots, n\}$.
- Let B be the set of length n bit-strings. That is, each element of B is a string of n-bits.
- The bits of an n-bit string are numbered from left to right starting with 1.

  For example, the numbering of a 5-bit string is

$$b_1 b_2 b_3 b_4 b_5.$$

- Consider the function $f : P(A) \to B$ given as

$$f(X) = b_1 \ldots b_n$$

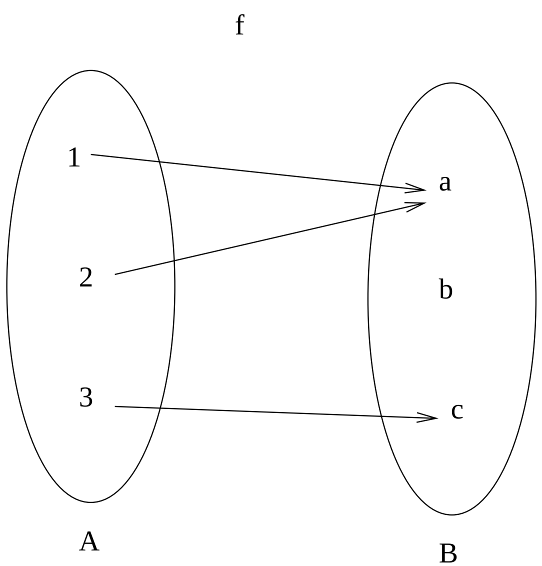
with $b_i = 1$ if and only if $i \in X$.

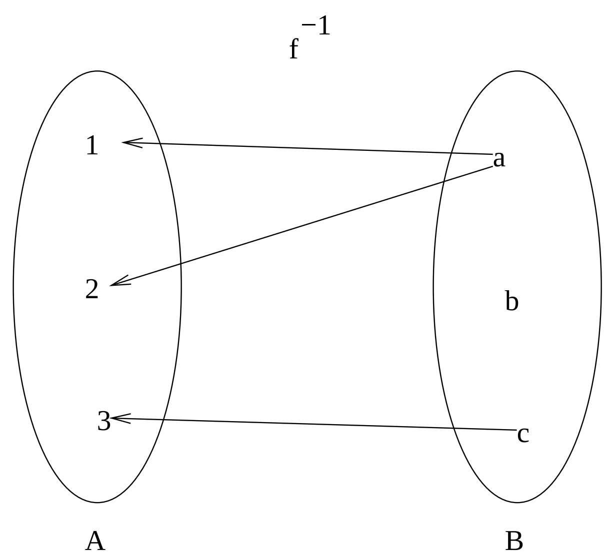
- Clearly, f is a 1-1 correspondence.

- For example, with $n = 3$, we have

X	$f(X)$
$\{\}$	000
$\{1\}$	100
$\{2\}$	010
$\{3\}$	001
$\{2,3\}$	011
$\{1,3\}$	101
$\{1,2\}$	110
A	111

Illustration — Is Inverse a Function?





Inverse Functions

1. Let $f : A \rightarrow B$.
2. So f is a relation from A to B.
3. That is, $f \subseteq A \times B$.
4. So $f^{-1} \subseteq B \times A$ is a relation.
5. Is f^{-1} a function from B to A?

Inverse Functions: Is every element in the co-domain related to at least one element in the domain?

1. $\forall y \in B$, $\exists x \in A$ such that $(y, x) \in f^{-1}$?
2. $(y, x) \in f^{-1}$ if and only if $(x, y) \in f$.
3. That is, $\forall y \in B$, $\exists x \in A$ such that $f(x) = y$?
4. The answer is yes if and only if f is onto.
5. That is, every element in the co-domain is related to at least one element in the domain if and only if the function is onto.

Inverse Functions: Is every element in the co-domain related to at most one element in the domain?

1. $\forall (y, x), (y, x') \in f^{-1}$, $x = x'$?

2. $\forall (x, y), (x', y) \in f$, $x = x'$?

3. The answer is yes if and only if f is 1-1.

4. That is, every element in the co-domain is related to at most one element in the domain if and only if the function is 1-1.

Inverse Functions

Theorem 7.2.1. If $f : A \to B$ is a function, then $f^{-1} \subseteq B \times A$ is a relation from B to A. If $f : A \to B$ is a 1-1 correspondence, then $f^{-1} \subseteq B \times A$ is a function from B to A.

Conversely, If $f^{-1} \subseteq B \times A$ is a function from B to A, then $f : A \to B$ is a 1-1 correspondence.

That is, $f^{-1} : B \to A$ if and only if f is a 1-1 correspondence.

Inverse Functions are 1-1 Correspondences

Theorem 7.2.2. Let $f : A \to B$ and $f^{-1} : B \to A$. Then f^{-1} is a 1-1 correspondence.

Let $g = f^{-1}$.

By Theorem 7.2.1:

$$g^{-1} : A \to B \text{ if and only if } g \text{ is a 1-1 correspondence.}$$

But $g^{-1} = f$ and indeed f is a function, thus $g = f^{-1}$ is a 1-1 correspondence.

Function Composition

Let $f : A \to B$, $g : B \to C$. The functions f and g can be composed to become a function $gf : A \to C$ given by

$$\forall x \in A, (gf)(x) = g(f(x)).$$

Example 1: Function Composition

Let $f : \mathbf{Z} \to \mathbf{Z}$ be $f(x) = x + 1$ and let $g : \mathbf{Z} \to \mathbf{Z}$ be $g(x) = x^2$.

The composition $gf : \mathbf{Z} \to \mathbf{Z}$ is

$$(gf)(x) = g(f(x)) = g(x+1) = (x+1)^2.$$

Example 2: Function Composition

- Let $f : \{1, 2, 3\} \rightarrow \{a, b, c, d, e\}$ be

$$f = \{(1, c), (2, b), (3, a)\}$$

- Let $g : \{a, b, c, d, e\} \rightarrow \{x, y, z\}$ be

$$g = \{(a, y), (b, y), (c, z), (d, z), (e, z)\}.$$

- The composition $gf : \{1, 2, 3\} \rightarrow \{x, y, z\}$ is

$$(gf)(1) \quad = \quad g(f(1)) = g(c) = z,$$

$$\begin{aligned}(gf)(2) &= g(f(2)) = g(b) = y, \\ (gf)(3) &= g(f(3)) = g(a) = y.\end{aligned}$$

- That is,

$$gf = \{(1, z), (2, y), (3, y)\}.$$

Function Composition is Associative

Let $f : A \to B$, $g : B \to C$, $h : C \to D$. We have $h(gf), (hg)f : A \to D$ and

$$h(gf) = (hg)f$$

Proof: For any $x \in A$,

$$(h(gf))(x) = h((gf)(x)) = h(g(f(x)))$$

and

$$((hg)f)(x) = (hg)(f(x)) = h(g(f(x))).$$

Notes: Function Composition is Associative

- $(h(gf))(x)$: the value of the composition of h and gf at x.
- $h((gf)(x))$: the value of h at $(gf)(x)$.
- $((hg)f)(x)$: the value of the composition of hg and f at x.
- $(hg)(f(x))$: the value of hg at $f(x)$.
- $h(g(f(x)))$: the value of h at $g(f(x))$.

The Identity Function of a Set

- Let A be a set.
- The identity relation

$$I_A = \{(x, x) \in A \times A \mid x \in A\}$$

is clearly a function from A to A.

- Furthermore, I_A is a 1-1 correspondence from A to A.

Composition with Identity Functions

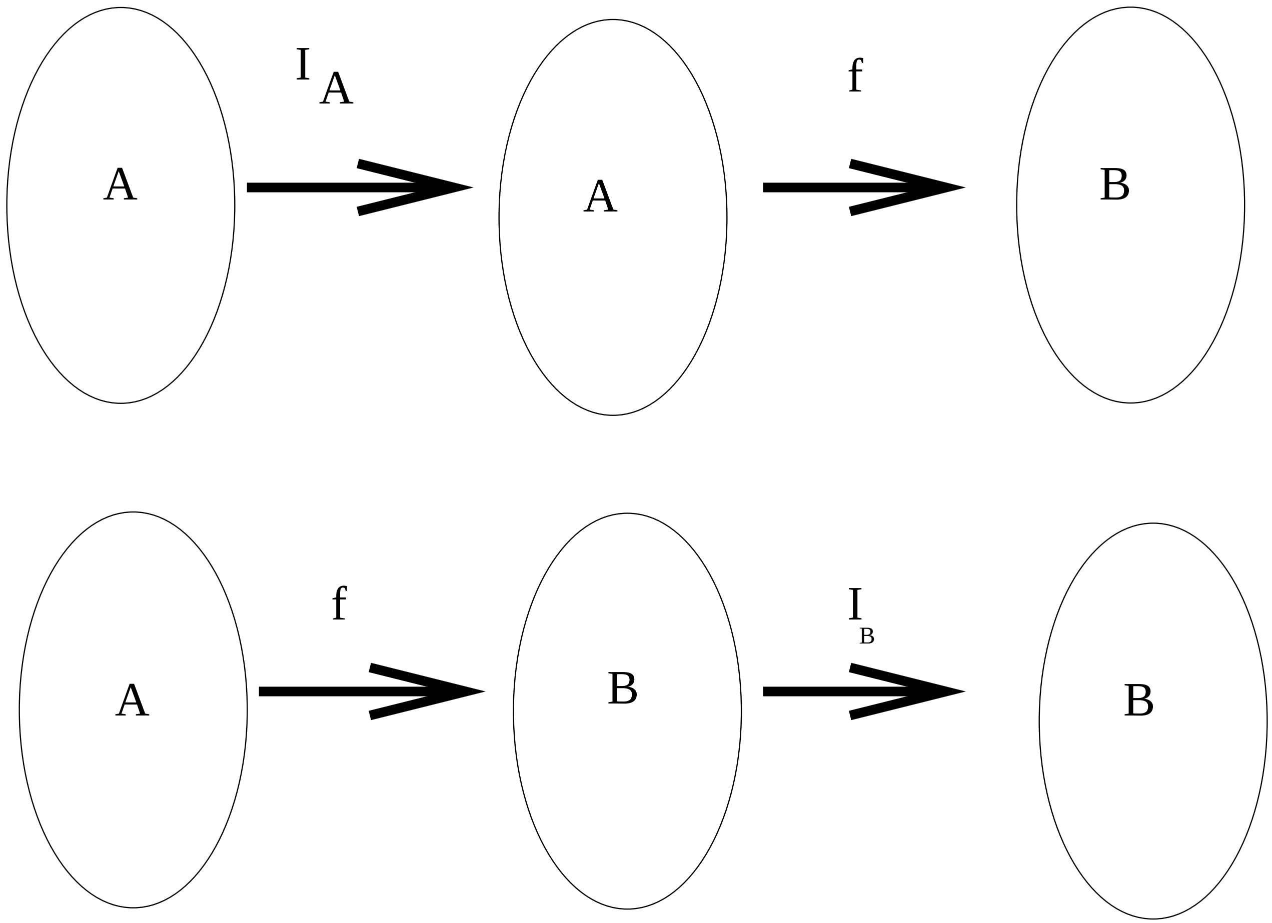
Theorem 7.4.1. Let $f : A \to B$, and I_A, I_B be the identity functions on A and B respectively. Then

$$fI_A = f,\ I_Bf = f.$$

Proof:

$$(fI_A)(x) = f(I_A(x)) = f(x) \quad (\because I_A(x) = x).$$

$$(I_Bf)(x) = I_B(f(x)) = f(x) \quad (\because I_B(y) = y).$$


A
I
A
A
f
B
A
f
B
I
B
B

Composition with the Inverse

Theorem 7.4.2. Let $f : A \to B$ be a bijection. Then

$$f^{-1}f = I_A, \quad ff^{-1} = I_B.$$

Proof:

Note that the compositions are formed as follows:

$$A \xrightarrow{f} B \xrightarrow{f^{-1}} A,$$

$$B \xrightarrow{f^{-1}} A \xrightarrow{f} B.$$

Proof: $f^{-1}f = I_A$

- For any $x \in A$,
$$(f^{-1}f)(x) = f^{-1}(f(x)).$$

- Let $f(x) = y$. Then
$$x = f^{-1}(y).$$

- Thus, for any $x \in A$,
$$(f^{-1}f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x = I_A(x).$$

- That is,
$$f^{-1}f = I_A.$$

Proof: $ff^{-1} = I_B$

- For any $y \in B$,
$$(ff^{-1})(y) = f(f^{-1}(y))$$
- Let $f^{-1}(y) = x$. Then
$$f(x) = y.$$
- Thus, for any $y \in B$,
$$(ff^{-1})(y) = f(f^{-1}(y)) = f(x) = y = I_B(y).$$
- That is,
$$ff^{-1} = I_B.$$

Composition One-to-One Functions

Theorem 7.4.3. If $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are one-to-one, then $gf : A \longrightarrow C$ is one-to-one.

Proof:

For any $x, y \in A$, let

$$(gf)(x) = (gf)(y).$$

Then

$$g(f(x)) = g(f(y)).$$

Since g is 1-1, so

$$f(x) = f(y).$$

Since f is 1-1, so

$$x = y.$$

That is, gf is 1-1.

Composition Onto Functions

Theorem 7.4.4. If $f : A \to B$ and $g : B \to C$ are onto, then $gf : A \to C$ is onto.

Proof:

For any $z \in C$, since g is onto, there is $y \in B$ such that

$$z = g(y).$$

Since f is onto, there is $x \in A$ such that

$$y = f(x).$$

Combining, we have

$$z = g(y) = g(f(x)) = (gf)(x).$$

Thus, for any $z \in C$, there is $x \in A$ such that $z = (gf)(x)$. That is, gf is onto.

Composition of 1-1 Correspondences

The composition of two 1-1 correspondences is a 1-1 correspondence.

Proof: The composition of two 1-1 functions is 1-1. The composition of two onto functions is onto. Since a 1-1 correspondence is both 1-1 and onto, the result follows.