Chapter 4: Sequences and Mathematical Induction

- Chapter 4 discusses three principles:
  1. the principle of (ordinary) mathematical induction,
  2. the principle of strong mathematical induction, and
  3. the well-ordering principle for the integers.
- The corresponding methods of proof are illustrated.
- The chapter also introduces finite and infinite sequences, the summation, the product, and the factorial notations.

Infinite Sequences

An **infinite sequence** is an ordered list of (usually) real numbers and is (usually) denoted

$$a_1, a_2, a_3, \ldots$$

or

$$a_0, a_1, a_2, \ldots$$

Each individual element a_i is called a **term**. The integer i is called the **index** of the term a_i.

Finite Sequences

A **finite sequence** of n elements is an ordered list of (usually) real numbers and is (usually) denoted

$$a_1, a_2, \ldots, a_n$$

or

$$a_0, a_1, \ldots, a_{n-1}$$

or even

$$a_m, a_{m+1}, \ldots, a_{m+n-1}$$

where $m \in \mathbf{Z}$.

The first and the last terms are also known as the **initial** and the **final** terms respectively.

Term Formulas

A finite sequence can be given by listing its terms. But it is impossible to list all the terms of an infinite sequence.

The terms of a sequence can also be given by a formula that tells what the term a_i is for index i.

For example, the sequence $a_i = \frac{(-1)^i}{i^2}$ for $i = 1$ to $i = 5$ is

$$-\frac{1}{1}, \quad \frac{1}{4}, \quad -\frac{1}{9}, \quad \frac{1}{16}, \quad -\frac{1}{25}.$$

Index Names

In a term formula any symbol could have been used for the index without changing the sequence.

For example, the same sequence could have been given as

$$a_j = \frac{(-1)^j}{j^2},\ j = 1, \ldots, 5; \quad \text{or} \quad a_k = \frac{(-1)^k}{k^2},\ k = 1, \ldots, 5.$$

(Recall the dummy variables in an integral.)

The Summation Notation

The sum of the terms of a sequence is most conveniently given by the summation notation. For example:

$$\begin{aligned}
\sum_{i=1}^{\infty} a_i &= a_1 + a_2 + a_3 + \cdots \\
\sum_{i=1}^{n} b_i &= b_1 + \cdots + b_n \\
\sum_{j=3}^{7} c_j &= c_3 + c_4 + c_5 + c_6 + c_7
\end{aligned}$$

$$\sum_{k=1}^{5} \frac{(-1)^k}{k} \quad = \quad -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5}$$

Telescoping Sums

A summation in which adjacent terms cancel each other partially such that only part of the initial (first) term and part of the final (last) term remain is called a **telescoping sum**.

For example, consider

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}.$$

The sum can be obtained by writing $\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$ and we have

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n}(\frac{1}{i} - \frac{1}{i+1}) = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1}$$

The Product Notation

The product of the finite sequence

$$a_m, a_{m+1}, \ldots, a_n$$

is written

$$\prod_{i=m}^{n} a_i = a_m \times \cdots \times a_n$$

where $m \leq n$.

For example, with $m \leq n$,

$$\prod_{i=m}^{n} \frac{i}{i+1} = \frac{m}{m+1} \times \frac{m+1}{m+2} \times \cdots \times \frac{n}{n+1} = \frac{m}{n+1}.$$

The Factorial Notation

In particular, the product of the first n positive integers is

$$\prod_{i=1}^{n} i = 1 \cdot 2 \cdot \cdots \cdot n = n!.$$

We adopt the convention that

$$0! = 1.$$

Some Properties of Summations and Products

Let $m \leq n$. We have

$$\sum_{i=m}^{n}(\alpha a_i + \beta b_i) = \alpha \sum_{i=m}^{n} a_i + \beta \sum_{i=m}^{n} b_i$$

and

$$\prod_{i=m}^{n} a_i \prod_{i=m}^{n} b_i = \prod_{i=m}^{n} (a_i b_i).$$

Change of Variables

Sometimes it is useful to change the summation indices to simplify the summations.

For example, consider

$$\sum_{i=0}^{n-1} i + \sum_{j=1}^{n} (j-1).$$

If we let $k = j - 1$, then $k = 0$ when $j = 1$ and $k = n - 1$ when $j = n$. Thus the sum becomes

$$\sum_{i=0}^{n-1} i + \sum_{k=0}^{n-1} k = \sum_{i=0}^{n-1} i + \sum_{i=0}^{n-1} i = 2\sum_{i=0}^{n-1} i = 2\sum_{i=1}^{n-1} i.$$

Principle of Mathematical Induction

Let $P(n)$ be a property that is defined for integers n, and let N be a fixed integer. Suppose the following two properties hold:

1. $P(N)$ is true.

2. For all integers $n \geq N$, if $P(n)$ is true then $P(n+1)$ is true.

Then the statement

for all integers $n \geq N, P(n)$

is true.

Why?

This can be seen as follows. $P(N)$ and the second property implies $P(N+1)$ is true. $P(N+1)$ and the second property implies $P(N+2)$ is true. Etc.

Principle of Mathematical Induction (The Usual Form)

Let $P(n)$ be a property that is defined for positive integers n. Suppose the following two properties hold:

1. $P(1)$ is true.

2. For all integers $n \geq 1$, if $P(n)$ is true then $P(n+1)$ is true.

Then the statement

$$\forall n \in \mathbf{Z}^{+}, P(n)$$

is true.

The Inductive Property of the Integers

Suppose S is a set of integers satisfying the two properties (1) $N \in S$, and (2) for all integers $k \geq N$, if $k \in S$ then $k+1 \in S$. Then S must contain every integer greater than or equal to N.

This can be seen as follows. $N \in S$ and the second property implies $N+1 \in S$. $N+1 \in S$ and the second property implies $N+2 \in S$. $N+2 \in S$ and the second property implies $N+3 \in S$. Etc.

The Principle of Mathematical Induction Implies the Inductive Property of the Integers

Suppose the principle of mathematical induction is true and the set S is such that (1) $N \in S$, and (2) $\forall n \geq N, (n \in S) \rightarrow (n+1 \in S)$. We need to show that

$$\{N, N+1, N+2, N+3, \cdots\} \subseteq S.$$

Proof: Let $P(n) =$ "$n \in S$".

By (1), $P(N)$ is true.

Suppose $P(n)$ is true for some $n \geq N$. By definition, $n \in S$. By (2), $n+1 \in S$. This means $P(n+1)$ is true. Thus $P(n) \rightarrow P(n+1)$ when $n \geq N$.

By the principle of mathematical induction, $P(n)$ is true for all $n \geq N$. That is, $n \in S$ is true for all $n \geq N$ and the claim is proved.

The Inductive Property of the Integers Implies the Principle of Mathematical Induction

Suppose the inductive property of integer is true and the statement $P(n)$ is such that (1) $P(N)$ is true, and (2) $\forall n \geq N, P(n) \rightarrow P(n+1)$. We need to show that $P(n)$ is true for all $n \geq N$.

Proof: Let

$$S = \{n \geq N |\ P(n)\}.$$

By (1), $N \in S$.

Let $n \in S$ for some $n \geq N$. By the definition of S, $P(n)$ is true. By (2) we have $P(n+1)$ is true and so $n+1 \in S$. Thus for any $n \geq N$, $n \in S$

implies $n+1 \in S$. By the inductive property of integers we know

$$\{N, N+1, N+2, \ldots\} \subseteq S.$$

That is, for all $n \geq N$, $P(n)$ is true.

The Equivalence of the Principle of Mathematical Induction and the Inductive Property of the Integers

It is important to emphasize that we prove neither the principle of mathematical induction nor the inductive property of the integers. These cannot be proved. They are so basic that we take them as axioms.

So what did we prove? We proved that these two axioms are logically equivalent.

The Method of Proof by Mathematical Induction

To prove the statement

for all integers $n \geq N$, $P(n)$

by mathematical induction we have to establish two properties:

(Basis Step.) Show that $P(N)$ is true.

(Inductive Step.) Show that for any $n \geq N$, if $P(n)$ then $P(n+1)$.

If the two properties hold, then by the principle of mathematical induction, $P(n)$ is true for all $n \geq N$.

The assumption "$P(n)$" (is true) is called the **inductive hypothesis**.

Example: Proposition 4.2.1

Proposition 4.2.1: Let $P(n)$ be the statement

$$\exists \text{ non-negative integers } x, y \text{ such that } n = 3x + 5y.$$

Then $P(n)$ is true for $n \geq 8$.

Proof:

Since $8 = 3 \cdot 1 + 5 \cdot 1$, so $P(8)$ is true. (The basis step is proved.)

Let $P(n)$, for some $n \geq 8$, be true. (This is the inductive hypothesis.) Then there are $x, y \in \mathbf{Z}^+$ such that

$$n = 3x + 5y.$$

There are two cases to consider: $y \geq 1$ or $y = 0$.

If $y \geq 1$, then

$$n + 1 = 3(x + 2) + 5(y - 1)$$

and $P(n + 1)$ is true.

If $y = 0$, then $x \geq 3$ (why?) and we have

$$n + 1 = 3(x - 3) + 5(y + 2)$$

and $P(n + 1)$ is also true.

Thus the inductive step is proved.

By the principle of mathematical induction, the proposition is proved.

Example: Theorem 4.2.2

Theorem 4.2.2: For all integers $n \geq 1$,

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Proof:

When $n = 1$, l.h.s = 1 and r.h.s = $\frac{1(1+1)}{2} = 1$. Thus the theorem is true when $n = 1$. (This is the basis step.)

Suppose the theorem is true for some $n \geq 1$. (This is the inductive hypothesis.) Then we have

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Add $n+1$ to both sides, the equation becomes

$$\sum_{i=1}^{n+1} i = \frac{n(n+1)}{2} + n + 1 = \frac{(n+1)(n+2)}{2}.$$

This shows that the theorem is true at $n+1$. (This is the inductive step.)

By the principle of mathematical induction, the theorem is true for all integers $n \geq 1$.

Example: Sum of a Geometric Sequence

Theorem 4.2.3: For all integers $n \geq 0$ and all real numbers $r \neq 1$:

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}.$$

Proof: When $n = 0$, l.h.s $= 1$ and r.h.s $= \frac{r-1}{r-1} = 1$. Thus the formula is true when $n = 0$. (This proves the basis step.)

Now assume for some $n \geq 0$, we have

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}.$$

Then

$$\sum_{i=0}^{n+1} r^i = \frac{r^{n+1}-1}{r-1} + r^{n+1} = \frac{r^{n+2}-1}{r-1}.$$

Thus if the formula is true at an $n \geq 0$ then it is also true at $n+1$.

By the principle of mathematical, the formula is true for all $n \geq 0$.

Example: Divisibility

Proposition 4.3.1. For all integers $n \geq 1$. $2^{2n} - 1$ is divisible by 3.

Proof:

Since $2^{2\times 1} - 1 = 3$ is divisible by 3, so the proposition is true when $n = 1$. (This is the basis step.)

Assume $2^{2n} - 1$ is divisible by 3 for an $n \geq 1$.

$$\begin{aligned} 2^{2(n+1)} - 1 &= 2^{2n} \times 4 - 1 \\ &= 3 \times 2^{2n} + 2^{2n} - 1 \end{aligned}$$

Thus $2^{2(n+1)} - 1$ is divisible by 3 by the inductive hypothesis. (The sum of multiples of 3 is a multiple of 3.)

By the principle of mathematical induction, the proposition is proved.

The Principle of Strong Mathematical Induction

Let $P(n)$ be a property that is defined for integers n, and let $N \leq M$ be fixed integers. Suppose the following two properties are true:

1. $P(N), \ldots, P(M)$ are all true. (The basis step.)
2. For all integers $n \geq M$, if $P(k)$ is true for all integers k with $N \leq k \leq n$, then $P(n+1)$ is true. (The inductive step.)

Then the statement "for all integers $n \geq N, P(n)$" is true.
(The supposition that $P(k)$ is true for all integers k with $N \leq k \leq n$ is called the induction hypothesis.)

The Principle of Strong Mathematical Induction (The Usual Form)

Let $P(n)$ be a property that is defined for positive integers n. Suppose the following two properties are true:

1. $P(1)$ is true. (The basis step.)
2. For all integers $n \geq 1$, if $P(k)$ is true for all integers k with $1 \leq k \leq n$, then $P(n+1)$ is true. (The inductive step.)

Then the statement "$\forall n \in \mathbf{Z}^+, P(n)$" is true.

Ordinary versus Strong Mathematical Induction

Note that in ordinary mathematical induction, to move one step we need only the preceding step to be true.

But in strong mathematical induction, to move one step we need all the preceding steps to be true.

However, despite this apparent difference, the methods of ordiniary and strong mathematical induction are equivalent in the sense that whatever one method can prove, the other method can prove too. But one method may be more convenient than the other.

Example: Divisibility by a Prime

Let $P(n)$ be "n is divisible by a prime". Show that $P(n)$ is true for all integers $n \geq 2$.

Proof:The basis step is to prove $P(2)$. Since 2 is a prime so $P(2)$ is true.

Assume that $P(2)$, $\cdots$, $P(n)$ are true. (This is the inductive hypothesis.) Consider $P(n+1)$.

If $n+1$ is a prime then $P(n+1)$ is obviously true. Otherwise, there are positive integers $a > 1$ and $b > 1$ such that $n+1 = ab$. Clearly $2 \leq a < n+1$ so $P(a)$ is true. The prime divisor of a is also a prime divisor of $n+1$. Thus $P(n+1)$ is true.

By the principle of strong mathematical induction, $P(n)$ is true for all $n \geq 2$.

The Well-Ordering Principle for the Integers

Let S be a non-empty set of integers all of which are greater than some fixed integer. The S has a least element. That is,

$$\exists l \in S \text{ such that } \forall x \in S, l \leq x.$$

Note that the well-ordering principle for integers is equivalent to the ordinary and strong principles of mathematical induction.

Why?

Note that S is non-empty and all its members are greater than the integer L. Pick any $n \in S$. One of the integers

$$L + 1, \ldots, n$$

is the least integer of S.

Examples

1. Consider $\{-3, 0, 4, 7\}$. Clearly -3 is the least element of the given set.

2. Consider $\mathbf{Z}^+$. Every element of $\mathbf{Z}^+$ is greater than 0. By the well-ordering principle, $\mathbf{Z}^+$ should have a least element. Indeed $1 \in \mathbf{Z}^+$ is the least element of $\mathbf{Z}^+$.

3. Consider $\mathbf{Z}$. Clearly there are no least elements.

The Quotient-Remainder Theorem (Existence Part)

Given any integer n and any positive integer d, there exist integers q and r such that

$$n = dq + r, \quad 0 \leq r < d.$$

Proof:Let

$$S = \{n - dk \mid n - dk \geq 0, k \in \mathbf{Z}\}.$$

First we must show that S is not empty. If $n \geq 0$, then

$$n - d \cdot 0 = n \geq 0$$

and $n \in S$. If $n < 0$, then

$$n - d \cdot n = n(1 - d) \geq 0$$

and $n - dn \in S$.

It follows from well-ordering principle that S contains a least element r. That is, exists $q \in \mathbf{Z}$ such that

$$n - dq = r$$

and thus

$$n = dq + r.$$

If $r \geq d$, then

$$n - d(q + 1) = r - d \geq 0$$

so $r - d \in S$ and r is not the least element of S. This is a contradiction so it has to be that $r < d$. Clearly $0 \leq r$ because $r \in S$.