The Pigeonhole Principle, Week 13 (Nov 13-17)

1. (Textbook Section 7.3, Q34.)
   Given a set of 52 distinct integers, show that there must be 2 whose sum or difference is divisible by 100.

2. (Textbook Section 7.3, Q35.)
   Show that if 101 integers are chosen from 1 to 200 inclusive, there must be 2 with the property that one is divisible by the other.

3. (Textbook Section 7.3, Q36)

   (a) Suppose $a_1, a_2, \cdots, a_n$ is a sequence of n integers none of which is divisible by n. Show that at least one of the differences $a_i - a_j$ (for $i \neq j$) must be divisible by n.

   (b) Show that every finite sequence $x_1, x_2, \cdots, x_n$ of n integers has a consecutive subsequence $x_{i+1}, x_{i+2}, \cdots, x_j$ whose sum is divisible by n. (For instance, the sequence 3, 4, 17, 7, 16 has the consecutive subsequence 17, 7, 16 whose sum is divisible by 5.)

4. (Textbook Section 7.3, Q37)
   Observe that the sequence 12, 15, 8, 13, 7, 18, 19, 11, 14, 10 has three increasing subsequences of length four : 12, 15, 18, 19; 12, 13, 18, 19; and 8, 13, 18, 19. It also has one decreasing subsequence of length four: 15, 13, 11, 10. Show that in any sequence of $n^2 + 1$ distinct real numbers, there must be a sequence of length $n + 1$ that is either strictly increasing ot strictly decreasing.

5. (Textbook Section 7.3, Q38)
   What is the largest number of elements that a set of integers from 1 through 100 can have so that no one element in the set is divisible by another?
   (*Hint:* Imagine writing all the numbers from 1 through 100 in the form $2^k \cdot m$, where $k \geq 0$ and m is odd.)