CS3234: Logic and Formal Systems
Assignment 4, due 11:00 AM, October 7.

1. (10 marks) **Note updated problem statement**

   Consider the encoding of tournament scheduling in SAT presented in Lecture 7 (B/W slide 11). Encode the following constraints as propositional formulas. You may introduce propositional atoms other than $p_{x,y,z}$. In this case, formulate constraints between the new variables and $p_{x,y,z}$, also as propositional formulas. For example, if you introduce a variable $b_{1,4}$, which encodes whether Team 1 has a bye in Date 4, you need the following formula that expresses the connection between $b_{1,4}$ and the p variables:

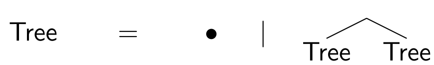
   $$\begin{aligned} (b_{1,4} \quad \rightarrow \quad & \neg p_{1,2,4} \wedge \neg p_{1,3,4} \wedge \cdots \wedge \neg p_{1,9,4} \wedge \\ & \neg p_{2,1,4} \wedge \neg p_{3,1,4} \wedge \cdots \wedge \neg p_{9,1,4}) \wedge \\ (\neg p_{1,2,4} \wedge \neg p_{1,3,4} \wedge \cdots \wedge \neg p_{1,9,4} \wedge & \\ \neg p_{2,1,4} \wedge \neg p_{3,1,4} \wedge \cdots \wedge \neg p_{9,1,4} \quad \rightarrow \quad & b_{1,4}) \end{aligned}$$

   As in the previous formula, you may use the $\cdots$ notation, if the meaning is clear.

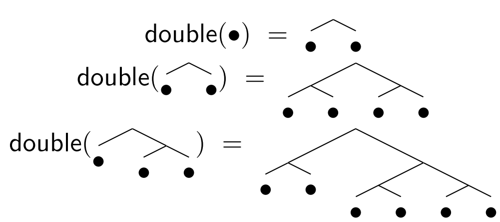
   - UNC (Team 1) plays Duke (Team 2) in the last date and in Date 11.
   - The following pairings must occur at least once in Dates 11 to 18: Duke (Team 2) – GT (Team 3), Duke (Team 2) – Wake (Team 4), GT (Team 3) – UNC (Team 1), UNC (Team 1) – Wake (Team 4).
   - No team can play away on both last dates.
   - Dates 1 and 8 are mirrored. This means two teams play each other in Date 1, iff they play each other in Date 8.

2. (10 marks) (a.k.a. exercise 8 in the lecture notes)

   We inductively define the set of binary trees as follows:



   Define (paper and Coq) a function that doubles all of the leaves in a tree. For example, this function should behave as follows:



3. (10 marks) (a.k.a. exercise 10 in the lecture notes)

   Using the double function you defined for the previous problem, and the following definition for leaves:

$$\mathsf{leaves}(t) \quad \equiv \quad \begin{cases} 1 & \text{when t} = \bullet \\ \mathsf{leaves}(t_l) + \mathsf{leaves}(t_r) & \text{when } t = \widehat{t_l \quad t_r} \end{cases}$$

   Please prove:

$$\forall t \ \big(\mathsf{leaves}(\mathsf{double}(t)) \ = \ \mathsf{leaves}(t) + \mathsf{leaves}(t)\big)$$

4. (10 marks)

   Now we define the function nodes as follows:

$$\mathsf{nodes}(t) \quad \equiv \quad \begin{cases} 0 & \text{when t} = \bullet \\ 1 + \mathsf{nodes}(t_l) + \mathsf{nodes}(t_r) & \text{when } t = \widehat{t_l \quad t_r} \end{cases}$$

   Please prove:

$$\forall t \ \big(\mathsf{nodes}(\mathsf{double}(t)) \ = \ \mathsf{nodes}(t) + \mathsf{leaves}(t)\big)$$

5. (15 marks) (a.k.a. exercise 11 in the lecture notes)

   Suppose we attempt to define streams inductively via the rule

$$\frac{n : \mathsf{nat} \qquad s : \mathsf{stream}}{n \,@\, s \ : \ \mathsf{stream}} \ \mathsf{Strm}$$

   Prove that in that case, stream is empty; that is,

$$\neg\exists s : \mathsf{stream}(\top)$$

   Note that this is (deMorgan-) equivalent to:

$$\forall s : \mathsf{stream}(\bot)$$

   If you set this up correctly, the proof should be very short.

6. (15 marks) (a.k.a. exercise 5 in the lecture notes)

   Give a series of rules and a complete and invertible set of objects satisfying those rules that is **neither** the least (inductive) nor greatest (coinductive).

7. (20 marks)

   Please prove

$$\forall t \ \big((t = \widehat{t \quad t}) \ \Rightarrow \ \bot\big)$$

   You can assume that generators are injective; that is, from

$$\widehat{t_{11} \quad t_{12}} \ = \ \widehat{t_{21} \quad t_{22}}$$

you may conclude

$$t_{11} \;=\; t_{21} \qquad \text{and} \qquad t_{12} \;=\; t_{22}$$

You may also assume that

$$\forall t_1 \forall t_2 \big((\bullet \;=\; \overset{\frown}{t_1 \quad t_2}) \;\Rightarrow\; \bot \big)$$