CS3234 Logic and Formal Systems

Midterm Examination Questions

17/09/2009

This examination question booklet has 12 pages, including this cover page, and contains 14 questions. Answer all questions. You have 60 minutes to complete the examination.

The examination has two parts:

- Questions 1–12 are MCQ questions. Use a B2 pencil to fill up the provided MCQ form. In the MCQ form, leave Section A blank, and fill up Sections B and C.
- Questions 13 and 14 require a written answer. Use the space provided in this question booklet to answer these two questions.

Enter your matriculation number here:

After finishing, place the MCQ sheet and this question booklet on top of the question sheet and leave both on the table, when you exit the room.

Multiple Choice Part

Question 1: (5 marks) The shortest formula in propositional logic is

$$p$$

where p is a propositional atom. Which one of the following statements is true?

1 [A] The formula p is valid and satisfiable.

1 [B] The formula p is invalid and unsatisfiable.

1 [C] The formula p is invalid and satisfiable.

1 [D] The formula p is valid and unsatisfiable.

1 [E] None of the above.

Answer 1:

1 [C] The formula p evaluates to F under the evaluation $p \mapsto F$, and therefore the formula is invalid. It evaluates to T under the valuation $p \mapsto T$ and therefore the formula is satisifiable.

Question 2: (10 marks) Which one of the following statements is true?

2 [A] Some sentences in predicate calculus may be satisfied by one model, but not by another one.

2 [B] All sentences in predicate calculus are either satisfied by all models or by none.

2 [C] No sentence in predicate calculus is satisified by all models.

2 [D] Every sentence is satisfied by at least one model.

2 [E] None of the above.

Answer 2:

2 [A] Let $\mathcal{P} = \{P\}$ and let us say that P is a unary predicate. Further, let $\mathcal{F} = \{c\}$ and let

us say that c is a constant. The sentence $P(c)$ is satisfied in every model $\mathcal{M}$ in which $c^{\mathcal{M}} \in P^{\mathcal{M}}$, and unsatisified in every model $\mathcal{M}$ in which $c^{\mathcal{M}} \notin P^{\mathcal{M}}$.

Question 3: (10 marks) Which one of the following statements is **false**?

3 [A] There are sentences that are satisfied by only finite models.

3 [B] There are sentences that are satisfied by models of any size but not infinite ones.

3 [C] There are sentences that are satisfied by some infinite models.

3 [D] There are sentences that are are satisfied by only finite models of sizes ranging from 42 to 117.

3 [E] One of the above is false.

Answer 3:

3 [B] Theorem 2.25 (Löwenheim-Skolem Theorem) directly contradicts this statement.

[Note that the last alternative would be false, if none of the preceding alternatives were false. In that case, E would be the answer to choose, because E would be the only false statement.]

Question 4: (10 marks) Consider the following attempt to prove the sequent

$$\forall x P(x),\ \exists x Q(x) \vdash \forall y (P(y) \wedge Q(y))$$

1		$\forall x P(x)$	premise
2		$\exists x Q(x)$	premise
3	x_0	$P(x_0)$	$\forall e$ 1
4	x_0	$Q(x_0)$	assumption
5		$P(x_0) \wedge Q(x_0)$	$\wedge i$ 3,4
6		$P(x_0) \wedge Q(x_0)$	$\exists e$ 2, 4–5
7		$\forall y (P(y) \wedge Q(y))$	$\forall i$ 3–6

Which one of the following statements about this proof attempt is correct?

4 [A] This is not a correct proof because the variable y is introduced in Line 7, and not within a box.

4 [B] This is not a correct proof because Line 6 is inside a box, but uses Line 2, which is outside the box.

4 [C] This is not a correct proof because both Line 3 and Line 4 introduce the same variable x_0.

4 [D] This is not a correct proof because Line 2 must not use the same variable as Line 1; it should say $\exists z Q(z)$.

4 [E] None of the above. This proof is correct.

Answer 4:

4 [C] Variables introduced in nested boxes must be distinct. Otherwise, proofs can be constructed for invalid sequents. The given "proof" is a drastic example of this.

Question 5: (10 marks) Consider the following attempt to prove the sequent

$$(\forall x P(x)) \to Q,\ \forall x P(x) \vdash \forall z (P(z) \to Q)$$

1	$(\forall x P(x)) \to Q$	premise
2	$\forall x P(x)$	premise
3	x_0	
4	$P(x_0)$	$\forall e$ 2
5	Q	$\to e$ 1,2
6	$P(x_0) \to Q$	$\to i$ 4–5
7	$\forall z (P(z) \to Q)$	$\forall i$ 3–6

Which one of the following statements about this proof attempt is correct?

5 [A] This is not a correct proof because the variable z is introduced in Line 7, and not within a box.

5 [B] This is not a correct proof because Line 5 is inside a box, but uses Lines 1 and 2, which are outside the box.

5 [C] This is not a correct proof because Line 3 does not have a formula.

5 [D] This is not a correct proof because Line 4 uses Line 2, which is not in the immediately enclosing box.

5 [E] None of the above. This proof is correct.

Answer 5:

5 [E] The proof is correct.

Question 6: (10 marks) A formula in propositional logic is called *valid* if it evaluates to T for all valuations of its propositional atoms. A sentence in predicate logic is called *valid* if it is satisfied by all models.
Which one of the following statements holds?

6 [A] There is a linear time translation from propositional logic formulas to predicate logic sentences that preserves validity.

6 [B] A validity preserving translation from propositional logic formulas to predicate logic sentences exists, but all such translations require exponential time in the worst case, w.r.t. the size of the formula.

6 [C] There cannot be a validity preserving translation from propositional logic formulas to predicate logic sentences.

6 [D] The problem of finding a validity preserving translation from propositional logic formulas to predicate logic sentences is undecidable.

6 [E] None of the above

Answer 6:

6 [A] Consider an arbitrary propositional formula ϕ in which say n propositional atoms occur. Let us call these atoms $p_1, \ldots, p_n$. In order to construct a corresponding formula in predicate logic, we use the set of predicate symbols $\mathcal{P} = \{IsTrue\}$, where *IsTrue* is a unary predicate, and the set of function symbols $\mathcal{F} = \{p_1, \ldots, p_n\}$, all of which are constants. The translation is as follows:

$$\begin{array}{rcll} translate(p) & = & IsTrue(p) & \text{if } p \text{ is an atom} \\ translate(\neg\phi) & = & \neg translate(\phi) & \\ translate(\phi \wedge \psi) & = & translate(\phi) \wedge translate(\psi) & \\ translate(\phi \vee \psi) & = & translate(\phi) \vee translate(\psi) & \\ translate(\phi \rightarrow \psi) & = & translate(\phi) \rightarrow translate(\psi) & \end{array}$$

You can prove via structural induction on ϕ that $translate(\phi)$ is valid if and only if ϕ is valid. Note that *translate* runs in linear time, because it visits every symbol in ϕ only once.

For Questions 7–10, consider the following statement of a theorem in Coq and the first tactic (intros) used in its proof.

```
Lemma MidtermProblem: forall P Q R,

  ((P = Q) \/ (P = R)) -> (R /\ Q) -> P.

Proof.

  intros.
```

At this point, Coq shows the following goals:

```
1 subgoal

P : Prop

Q : Prop

R : Prop

H : P = Q \/ P = R

H0 : R /\ Q

______________________________________(1/1)

P
```

Question 7: (5 marks) Which tactic will move the goal from the above to:

```
1 subgoal

P : Prop

Q : Prop

R : Prop

H : P = Q \/ P = R

H0 : R

H1 : Q

______________________________________(1/1)

P
```

7 [A] `split`

7 [B] `intros`

7 [C] `contradiction`

7 [D] `destruct H0`

7 [E] `apply H0`

Answer 7:

7 [D] Use the following Coq scipt to verify this and the following three answers:

```
Require Import Classical.

Lemma Quiz1Problem: forall P Q R,

 ((P = Q) \/ (P = R)) -> (R /\ Q) -> P.

Proof.

intros.
destruct H0.
destruct H.
```

```
rewrite H; trivial.
rewrite <- H in H0; trivial.
```

Question 8: (5 marks) Now the goal is to do disjunction-elimination on `H`. Which tactic will do this?

8 [A] `destruct H`

8 [B] `left in H`

8 [C] `apply H`

8 [D] `generalize (classic H)`

8 [E] `split`

Answer 8:

8 [A]

Question 9: (5 marks) Coq now reports the following:

```
2 subgoals

P : Prop

Q : Prop

R : Prop

H : P = Q

H0 : R

H1 : Q

______________________________________(1/2)

P



______________________________________(2/2)

P
```

What tactic combination will solve the first goal?

9 [A] `destruct H`

9 [B] `split`

9 [C] `apply H0`

9 [D] `rewrite H; trivial`

9 [E] `rewrite <- H; trivial`

Answer 9:

9 [D]

Question 10: (5 marks) Now Coq reports the following status:

```
1 subgoal

P : Prop

Q : Prop

R : Prop

H : P = R

H0 : R

H1 : Q

______________________________________(1/1)

P
```

What will solve this goal?

10 [A] `destruct H`

10 [B] `rewrite H in H0; trivial`

10 [C] `split`

10 [D] `apply H`

10 [E] `rewrite <- H in H0; trivial`

Answer 10:

10 [E]

Question 11: (10 marks) Consider the following definition of the addition and subtraction predicates defined on the usual representation of integers using `Zero` and `Succ`.

```
Add(Zero, n2) = n2

Add(Succ n1, n2) = Succ (Add (n1, n2))


Sub (Zero, n2) = Zero

Sub (n1, Zero) = n1

Sub (Succ n1, Succ n2) = Sub (n1, n2).
```

Does the following formula

$$\forall n_1 \ \forall n_2 \ \texttt{Sub}(\texttt{Add}(n_1, n_2), n_1) = n_2$$

hold?

11 [A] Yes

11 [B] No

Answer 11:

11 [A] Excercise: Provide a proof for the formula using Coq!

Question 12: (10 marks) Using the same definition of the addition and subtraction predicates as in the previous question, does the following formula

$$\forall n_1 \ \forall n_2 \ \texttt{Add}(\texttt{Sub}(n_1, n_2), n_2) = n_1$$

hold?

12 [A] Yes

12 [B] No

Answer 12:

12 [B] Consider $n_1 = \texttt{Zero}$ and $n_2 = \texttt{Succ Zero}$, we have

$$\texttt{Add}(\texttt{Sub}(n_1, n_2), n_2) = \texttt{Add}(\texttt{Sub}(\texttt{Zero}, n_2 = \texttt{Succ Zero}), n_2 = \texttt{Succ Zero}) = \texttt{Succ Zero}$$

which is different from $n_1 = n_2 = \texttt{Zero}$.

Written Part

Question 13: (20 marks) Prove the validity of the following propositional logic sequent using natural deduction.

$$(p \to (q \lor r)) \vdash \neg q \to (p \to r).$$

[use the space below to write down the proof]

Answer 13:

1	$p \to (q \lor r)$	premise
2	$\neg q$	assumption
3	p	assumption
4	$q \lor r$	$\to e$ 3,1
5	q	assumption
6	$\bot$	$\neg e$ 5,2
7	r	$\bot e$ 6
8	r	assumption
9	r	$\lor e$ 4, 5–7, 8–8
10	$p \to r$	$\to i$ 3–9
11	$\neg q \to (p \to r)$	$\to i$ 2–10

Question 14: (40 marks) One of the following sequents in predicate logic is valid, and the other is invalid. For the valid one, provide a proof (20 marks); for the invalid one, provide a model as counter example (20 marks).

1.
$$\forall x\ \exists y\ P(x,y) \vdash \exists y \forall x\ P(x,y)$$

2.
$$\exists y\ \forall x\ P(x,y) \vdash \forall x \exists y\ P(x,y)$$

[use the space below to write down the proof and the model]

Answer 14: The first sequent is not valid. One counter-example is a model $\mathcal{M}$ whose universe has two elements, say a and b, such that $P^{\mathcal{M}} = \{(a,a),(b,b)\}$. Clearly $\forall x \exists\ P(x,y)$ holds in this model, but $\exists y \forall x\ P(x,y)$ does not hold.

The second sequent
$$\exists y\ \forall x\ P(x,y) \vdash \forall x \exists y\ P(x,y)$$
is valid, and here is a proof:

1		$\exists y \forall x P(x,y)$	premise
2	y_0	$\forall x\ P(x,y_0)$	assumption
3	x_0	$P(x_o,y_0)$	$\forall x\ e$ 2
4		$\exists y\ P(x_0,y)$	$\exists x\ i$ 3
5		$\forall x \exists y\ P(x,y)$	$\forall x\ i$ 3–4
6		$\forall x \exists y\ P(x,y)$	$\exists x\ e$ 1, 2–5