

CS3234 Logic and Formal Systems

Midterm Examination Questions

16/09/2010

This examination question booklet has 8 pages, including this cover page, and contains 12 questions, each worth 10 points. Use a 2B pencil to fill up the provided MCQ form. In the MCQ form, leave Section A blank, and fill up Sections B and C. For each question, choose only one answer! Choose the answer that *most accurately* answers the question.

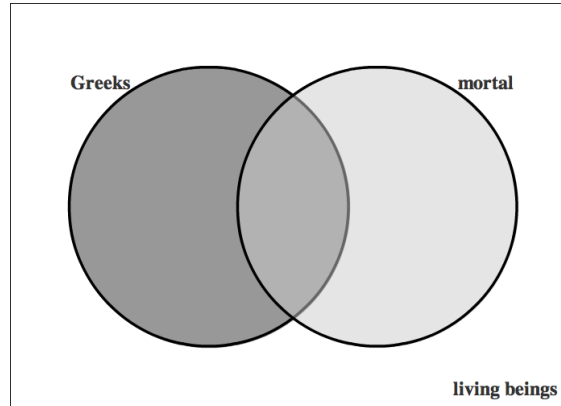
Enter your matriculation number here:

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After finishing and before you exit the room, place the MCQ sheet on top your table.

1 Term Logic

Question 1: (10 marks) Consider the following Venn diagram:



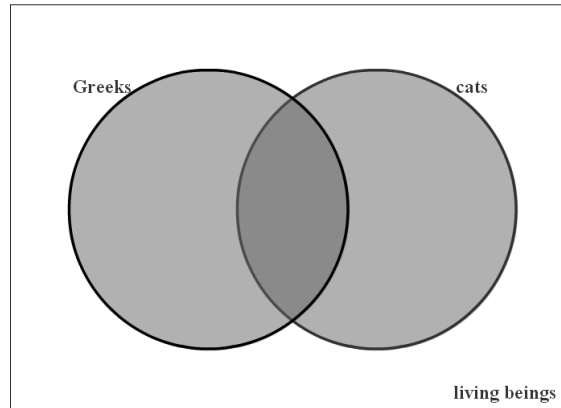
Which one of the following categorical propositions can be used in order to express that the dark grey area on the left of the diagram does not contain any elements?

- 1 **A** Some Greeks are not mortal
- 1 **B** Some Greeks are mortal
- 1 **C** All Greeks are mortal
- 1 **D** No Greeks are mortal
- 1 **E** Some non Greeks are non mortal

Answer 1:

- 1 **C** Look up the definition of All ... are ... in the notes.

Question 2: (10 marks) Consider the following Venn diagram:



Which one of the following categorical propositions can be used in order to express that the dark grey area in the center of the diagram does not contain any elements?

- 2 (A) Some Greeks are not cats
- 2 (B) Some Greeks are cats
- 2 (C) All Greeks are cats
- 2 (D) No Greeks are cats
- 2 (E) Some non Greeks are non cats

Answer 2:

- 2 (D) Look up the definition of No ... are ... in the notes.

Question 3: (10 marks)

Recall that in traditional logic, the set that a term refers to may be empty. Which one of the following alternatives is true?

- 3 **A** Both All x are y and No x are y may be valid in the same model.
- 3 **B** Both Some x are y and No x are y may be valid in the same model.
- 3 **C** Both All x are y and Some x are not y may be valid in the same model.
- 3 **D** None of $\{A, B, C\}$
- 3 **E** All of $\{A, B, C\}$

Answer 3:

- 3 **A** If in a particular model \mathcal{M} , the set $x^{\mathcal{M}}$ is empty, then All x are y and No x are y are both valid, according to the definition of their semantics.
None of the other alternatives are correct, regardless whether any of the set may be empty or not.

2 Propositional Logic

Question 4: (10 marks)

Recall the definitions of valid, satisfiable, and unsatisfiable. What is the relationship between the above concepts?

- 4 **A** If a formula ϕ is valid then ϕ is satisfiable; and if ϕ is invalid then ϕ is not satisfiable.
- 4 **B** If ϕ is valid then $\neg\phi$ is not satisfiable; and if ϕ is invalid then $\neg\phi$ is satisfiable.
- 4 **C** If $\neg\phi$ is not satisfiable, then ϕ is valid; and if $\neg\phi$ is satisfiable then ϕ is invalid.
- 4 **D** Two of $\{A, B, C\}$ are true.
- 4 **E** Three of $\{A, B, C\}$ are true.

Answer 4:

- 4 **D** The second part of the first alternative is not correct: A formula may be invalid, but

still satisfiable. This is the case when it evaluates to T for some valuations and to F for other valuations.

All other statements are correct. Thus, two of $\{A, B, C\}$ are true, namely B and C .

Question 5: (10 marks)

Recall that we say that a proof is *intuitionistic* when it does not use the double-negation-elimination rule ($\neg\neg e$), including derived rules like the law of excluded middle. On the other hand, if a proof **does** use double-negation-elimination, we say that the proof is *classical*.

There are two key points to make about intuitionistic logic:

1. Intuitionistic logic is **strictly** weaker than classical logic. In other words, there are theorems that are provable in classical logic that are not provable in an intuitionistic setting: in particular, the formulas $p \vee \neg p$ (using LEM) or $\neg\neg p \rightarrow p$ (using $\neg\neg e$).
2. As a consequence of item 1, *you cannot use the semantic method for classical propositional logic* (truth tables) to argue whether a formula has an intuitionistic proof.^a Instead, you will need to argue using natural deduction.

Consider the following formula in propositional logic:

$$\neg\neg\neg\neg p \rightarrow \neg\neg p$$

Please classify this formula as follows:

- 5 **A** Provable in intuitionistic logic, and provable in classical logic.
- 5 **B** Provable in intuitionistic logic, but not provable in classical logic.
- 5 **C** Not provable in intuitionistic logic, but provable in classical logic.
- 5 **D** Neither provable in intuitionistic logic nor classical logic.

^aWhen we cover *modal logic* later in the semester we will cover the semantics of intuitionistic logic.

Answer 5:

- 5 **A** Here is a proof in intuitionistic logic:

1	$\neg\neg\neg p$	assumption
2	$\neg p$	assumption
3	$\neg\neg p$	assumption
4	\perp	$\neg e$ 2, 3
5	$\neg\neg\neg p$	$\neg i$ 3-4
6	\perp	$\neg e$ 5,1
7	$\neg\neg p$	$\neg i$ 2-6
8	$\neg\neg\neg p \rightarrow \neg\neg p$	$\rightarrow i$ 1-7

Since everything true in intuitionistic logic is also true in classical logic, the answer must be (A).

Question 6: (10 marks)

Refer to the previous question for the difference between classical and intuitionistic proofs.

$$(\neg(\neg p \wedge \neg q)) \rightarrow (p \vee q)$$

Please classify this formula as follows:

- 6 (A) Provable in intuitionistic logic, and provable in classical logic.
- 6 (B) Provable in intuitionistic logic, but not provable in classical logic.
- 6 (C) Not provable in intuitionistic logic, but provable in classical logic.
- 6 (D) Neither provable in intuitionistic logic nor classical logic.

Answer 6:

- 6 (C) In classical logic, we can argue by natural deduction or by truth tables that this must be valid, so the answer must be (A) or (C). The tricky part is, which one? Suppose that a proof did exist in intuitionistic logic for this formula. Call that (hypothetical) proof $\Pi(P, Q)$; we will now use Π as a “subproof” to prove $P \vee \neg P$.

1	$(\neg(\neg p \wedge \neg q)) \rightarrow (p \vee \neg p)$	$\Pi(p, \neg p)$
2	$\neg p \wedge \neg\neg p$	assumption
3	$\neg p$	$\wedge e_1$ 2
4	$\neg\neg p$	$\wedge e_2$ 2
5	\perp	$\neg e$ 3, 4
6	$\neg(\neg p \wedge \neg\neg p)$	$\rightarrow i$ 2-5
7	$p \vee \neg p$	$\rightarrow e$ 1, 6

Notice here I have used the fact that any proof using P and Q can be adapted (using exactly the same steps) into a proof using P and $\neg P$. We can discuss why in tutorial if there in interest. **Conclusion:** since we have shown that a proof of the original formula can be used to prove $P \vee \neg P$, and since we know that $P \vee \neg P$ is not provable in intuitionistic logic, the original formula must **not** be provable in intuitionistic logic.

Question 7: (10 marks)

Refer above for the difference between classical and intuitionistic proofs.

$$(\neg\neg(p \vee \neg p) \rightarrow (p \vee \neg p)) \rightarrow (p \vee \neg p)$$

Please classify this formula as follows:

- 7 **A** Provable in intuitionistic logic, and provable in classical logic.
- 7 **B** Provable in intuitionistic logic, but not provable in classical logic.
- 7 **C** Not provable in intuitionistic logic, but provable in classical logic.
- 7 **D** Neither provable in intuitionistic logic nor classical logic.

Answer 7:

- 7 **A** Here is a proof in intuitionistic logic:

1	$\neg\neg(P \vee \neg P) \rightarrow (P \vee \neg P)$	assumption
2	$\neg(P \vee \neg P)$	assumption
3	P	assumption
4	$P \vee \neg P$	$\vee i$ 1, 3
5	\perp	$\neg e$ 4, 2
6	$\neg P$	$\neg i$ 3-5
7	$P \vee \neg P$	$\wedge i_2$ 6
8	\perp	$\neg e$ 7, 2
9	$\neg\neg(p \vee \neg p)$	$\neg i$ 2-8
10	$P \vee \neg P$	$\rightarrow e$ 1, 9
11	$(\neg\neg(P \vee \neg P) \rightarrow (P \vee \neg P)) \rightarrow (P \vee \neg P) \rightarrow i$ 1-10	

Since everything true in intuitionistic logic is also true in classical logic, the answer must be (A).

3 Predicate Logic

Question 8: (10 marks)

Consider the following formula in predicate logic:

$$\forall x \left(\forall x (P(x) \rightarrow \neg Q(x)) \rightarrow Q(x) \rightarrow \neg P(x) \right)$$

Classify this formula:

- 8 A Valid
- 8 B Invalid but satisfiable
- 8 C Not satisfiable

Answer 8:

- 8 A The formula is valid. Let us consider an arbitrary model \mathcal{M} and an arbitrary element x_0 in $U^{\mathcal{M}}$. Under the assumption of $Q(x)$ and $\forall y(P(y) \rightarrow \neg Q(y))$ (note that for clarity, we did variable renaming), the formula $P(x)$ cannot hold, since if it did, then due to the universally quantified implication, $\neg Q(x)$ would have to hold, contradicting the assumption of $Q(x)$. Thus, $\neg P(x)$ must hold.

Question 9: (10 marks)

Consider the following formula in predicate logic:

$$\exists x \left(\forall y (P(y) \rightarrow Q(x)) \right) \rightarrow \neg \exists x \left(\neg \exists y (P(x) \rightarrow Q(y)) \right)$$

Classify this formula:

- 9 A Valid
- 9 B Invalid but satisfiable
- 9 C Not satisfiable

Answer 9:

9 A We can transform the premise of the implication through renaming of x and y :

$$\exists y(\forall x(P(x) \rightarrow Q(y)))$$

and the conclusion of the implication through the known equivalences to:

$$\forall x(\exists y(P(x) \rightarrow Q(y)))$$

Now it is clear that if there exists a y (say y_0) such that the implication $P(x) \rightarrow Q(y)$ holds for all x , then for all x , there exists a y such that $P(x) \rightarrow Q(y)$ holds, namely the same y_0 .

Question 10: (10 marks)

Consider the following formula in predicate logic:

$$\exists x(\forall y(P(y) \rightarrow Q(x))) \rightarrow \exists x(P(x) \rightarrow Q(x))$$

Classify this formula:

10 A Valid

10 B Invalid but satisfiable

10 C Not satisfiable

Answer 10:

10 A The formula is valid. Assume there is an x such that for all y , we have $P(y) \rightarrow Q(x)$, and let us call it x_0 . Since $P(y) \rightarrow Q(x_0)$ holds for all y , it also holds for x_0 , and thus we can conclude $P(x_0) \rightarrow Q(x_0)$. Thus x_0 is a witness for $\exists x(P(x) \rightarrow Q(x))$. Overall, we can conclude

$$\exists x(\forall y(P(y) \rightarrow Q(x))) \rightarrow \exists x(P(x) \rightarrow Q(x))$$

More formally, we of course can use natural deduction to prove the validity of the formula, due to the correctness of deduction:

1	$\exists x(\forall y(P(y) \rightarrow Q(x)))$	assumption	
2	$\forall y(P(y) \rightarrow Q(x_0))$	assumption	x_0
3	$P(x_0) \rightarrow Q(x_0)$	$\forall x e 2$	
4	$\exists x(P(x) \rightarrow Q(x))$	$\exists x i 3$	
5	$\exists x(P(x) \rightarrow Q(x))$	$\exists x e 1, 2-4$	
6	$\exists x(\forall y(P(y) \rightarrow Q(x))) \rightarrow \exists x(P(x) \rightarrow Q(x))$	$\rightarrow \rightarrow i 1-5$	

Question 11: (10 marks)

Recall that we required a model for predicate logic to be non-empty. An alternative definition of models relaxes this requirement so that models are allowed to be empty. Given this possibility, consider the following formula in predicate logic:

$$\neg\exists x(P(x)) \wedge \forall xP(x)$$

Classify this formula:

- 11 **A** Valid if models can be empty, valid if models must be nonempty.
- 11 **B** Satisfiable-but-invalid if models can be empty; unsatisfiable if models must be nonempty.
- 11 **C** Satisfiable-but-invalid if models can be empty; satisfiable-but-invalid if models must be nonempty.
- 11 **D** Satisfiable-but-invalid if models can be empty; valid if models must be nonempty.
- 11 **E** Satisfiable-but-invalid if models can be empty; satisfiable-but-invalid if models must be nonempty.

Answer 11:

- 11 **B** It is enough that an empty model makes it satisfiable and that it is unsatisfiable when models must be nonempty. If the model is empty, then there does not exist an x , regardless of P , and for any x , $P(x)$ holds since there are not any x to begin with. Thus the empty model satisfies the formula. When the model is nonempty, then there is some element u in the model; if the right hand side of the conjunction is true, then $P(u)$ is true, so the left hand must be false; on the other hand, if the right hand is false then the conjunction is false to begin with.

Question 12: (10 marks)

Consider the following formula ϕ in predicate logic:

$$\forall z \left(Q(x) \wedge \forall x (P(z) \rightarrow R(x)) \wedge R(z) \rightarrow R(x) \right) \wedge P(x)$$

What is the result of the substitution $[x \Rightarrow f(x, y, z)]\phi$?

12 **A** $\forall z' \left(Q(f(x, y, z)) \wedge \forall x (P(z) \rightarrow R(f(x, y, z))) \wedge R(z') \rightarrow R(f(x, y, z)) \right) \wedge P(f(x, y, z))$

12 **B** $\forall z' \left(Q(f(x, y, z)) \wedge \forall x' (P(z') \rightarrow R(f(x, y, z))) \wedge R(z') \rightarrow R(f(x, y, z)) \right) \wedge P(f(x, y, z))$

12 **C** $\forall z \left(Q(f(x, y, z')) \wedge \forall x' (P(z) \rightarrow R(f(x', y, z'))) \wedge R(z') \rightarrow R(f(x, y, z')) \right) \wedge P(f(x, y, z))$

12 **D** $\forall z \left(Q(f(x, y, z)) \wedge \forall x (P(z) \rightarrow R(x)) \wedge R(z) \rightarrow R(f(x, y, z)) \right) \wedge P(f(x, y, z))$

12 **E** $\forall z' \left(Q(f(x, y, z)) \wedge \forall x (P(z') \rightarrow R(x)) \wedge R(z') \rightarrow R(f(x, y, z)) \right) \wedge P(f(x, y, z))$

Answer 12:

12 **E** This looks scary but it isn't so bad. Just avoid variable capture!