Predicate Logic II

CS 3234: Logic and Formal Systems

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September 9, 2010

Generated on Friday 10 September, 2010, 22:28

1 Proof Theory

1.1 Inheriting Natural Deduction of Propositional Logic

If we consider propositions as nullary predicates, propositional logic is a sub-language of predicate logic. It will come as no surprise that we can translate the rules for natural deduction in propositional logic directly to predicate logic. Each of the following rules applies to any formulas ϕ and ψ of predicate logic.

$$\frac{\phi \qquad \psi}{\phi \wedge \psi}[\wedge i] \qquad \frac{\phi \wedge \psi}{\phi}[\wedge e_1] \qquad \frac{\phi \wedge \psi}{\psi}[\wedge e_2]$$

$$\frac{\phi}{\phi \vee \psi}[\vee i_1] \qquad \frac{\psi}{\phi \vee \psi}[\vee i_2] \qquad \frac{\phi \vee \psi \qquad \boxed{\begin{array}{c}\phi \\ \vdots \\ \chi\end{array}} \qquad \boxed{\begin{array}{c}\psi \\ \vdots \\ \chi\end{array}}}{\chi}[\vee e]$$

$$\frac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \psi\end{array}}}{\phi \to \psi}[\to i] \qquad \frac{\phi \qquad \phi \to \psi}{\psi}[\to e]$$

$$\frac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \bot\end{array}}}{\neg\phi}[\neg i] \qquad \frac{\phi \qquad \neg\phi}{\bot}[\neg e]$$

$$\frac{\bot}{\phi}[\bot e] \qquad \frac{\neg\neg\phi}{\phi}[\neg\neg e]$$

1.2 Equality

We mentioned in "Semantics of Predicate Logic" that equality is usually interpreted to mean identity, which means that in a model $a =^{\mathcal{M}} b$ holds if and only if a and b are the same elements of the model's universe. It is safe to assume $t = t$ for any term t, because both sides of the equation will evaluate to the same element, regardless of the context (environment) in which we operate. The following equality introduction rule expresses this reasoning.

$$\frac{}{t = t}[= i]$$

The next rule, equality elimination, allows us to replace a term t_1 by another term t_2, provided that $t_1 = t_2$ is already proven. More precisely, in order to prove a formula ψ, in which a term t_2 appears (possibly multiple times), it is sufficient to prove $t_1 = t_2$ and the formula ψ' that results from ψ by replacing t_2 by t_1. The rule stated below uses a formula ϕ in which a free variable x represents the positions of t_2 in ψ, thus $\psi = [x \Rightarrow t_2]\phi$, and $\psi' = [x \Rightarrow t_1]\phi$.

$$\frac{t_1 = t_2 \qquad [x \Rightarrow t_1]\phi}{[x \Rightarrow t_2]\phi}\,[= e]$$

Using these two rules, we show: We show:

$$f(z) = g(z) \vdash h(g(z)) = h(f(z))$$

as follows:

1	$f(z) = g(z)$	premise
2	$h(f(z)) = h(f(z))$	$= i$
3	$h(g(z)) = h(f(z))$	$= e$ 1,2

Note that the formula $h(g(z)) = h(f(z))$ in Line 3 has the form $[x \Rightarrow t_2]\phi$, where t_2 is $g(z)$ and ϕ is $h(z) = h(f(z)$. If we use $f(z)$ for t_1, then Rule $= e$ asks us to prove $t_1 = t_2$ (Line 1), and $[x \Rightarrow t_1]\phi$ (Line 2).

1.3 Universal Quantification

Elimination of Universal Quantification Once you have proven $\forall x \phi$, you can replace x by any term t in ϕ, provided that t is free for x in ϕ, and thus "eliminate" the universal quantification.

$$\frac{\forall x \phi}{[x \Rightarrow t]\phi}\,[\forall x\ e]$$

This rule is justified by the semantics of $\forall x \phi$, since in a particular context (environment) any term t denotes a value in the model, and ϕ holds for all such values, if $\forall x \phi$ holds in the model.

In t any function symbols of the logic, as well as variables that are known in the context can be used.

Example 1. *We shall prove:* $S(g(john)), \forall x(S(x) \rightarrow \neg L(x)) \vdash \neg L(g(john))$

1	$S(g(john))$	*premise*
2	$\forall x(S(x) \rightarrow \neg L(x))$	*premise*
3	$S(g(john)) \rightarrow \neg L(g(john))$	$\forall x\ e$ *2*
4	$\neg L(g(john))$	$\rightarrow e$ *3,1*

Introduction of Universal Quantification The introduction rule for universal quantification is more complicated. Let us consider a new kind of box that allows us to introduce a fresh variable. For example,

$$\boxed{\vdots}^{\,z}$$

is a box in which the variable z can be used in terms, as in

$$\boxed{\begin{array}{c}\vdots\\ f(z) = f(z)\\ \vdots\end{array}}^{\,z}$$

Let us say we introduce a variable x_0 in a box. Without any assumptions on x_0, we prove a formula ψ, in which x_0 appears. The fact that x_0 appears in ψ, we can characterize by writing ψ as $[x \Rightarrow x_0]\phi$. Since we have not made any assumptions on x_0 within the box, we have shown that $[x \Rightarrow x_0]\phi$ holds for all possible instantiations of x by values of the universe; in other words, we can conclude $\forall x\phi$.

$$\frac{\boxed{\begin{array}{c}\vdots\\ [x \Rightarrow x_0]\phi\end{array}}^{\,x_0}}{\forall x\phi}\,[\forall x\ i]$$

The variable x_0 must be *fresh*; we cannot introduce the same variable twice in nested boxes. Freshness of course guarantees that x_0 is free for x in ϕ.

Example 2. *We shall prove the sequent* $\forall x(P(x) \to Q(x)), \forall x P(x) \vdash \forall x Q(x)$

1	$\forall x(P(x) \to Q(x))$	*premise*	
2	$\forall x P(x)$	*premise*	
3	$P(x_0) \to Q(x_0)$	$\forall x$ *e 1*	x_0
4	$P(x_0)$	$\forall x$ *e 2*	
5	$Q(x_0)$	$\to$ *e 3,4*	
6	$\forall x Q(x)$	$\forall x$ *i 3–5*	

1.4 Existential Quantification

Introduction of Existential Quantification For existential quantification, the easy direction is introduction.

$$\frac{[x \Rightarrow t]\phi}{\exists x\phi}\,[\exists x\ i]$$

In order to prove $\exists x\phi$, it suffices to find a term t as "witness", provided—as usual—that t is free for x in ϕ.

Example 3. *Assume that the set $\mathcal{F}$ contains a nullary function symbol c, and that the set $\mathcal{P}$ contains a unary predicate symbol P. We should be able to prove:*

$$\forall x P(x) \vdash \exists x P(x)$$

since at least the constant c should have the property P, once we know that all elements of the universe has the property P. The corresponding proof follows:

1	$\forall x P(x)$	*premise*
2	$[x \Rightarrow c]P(x)$	$\forall x$ *e 1*
3	$\exists x P(x)$	$\exists x$ *i 2*

Elimination of Existential Quantification Finally, for elimination of existential quantification, we combine the two kinds of boxes; we simultaneously introduce a fresh variable *and* an assumption.

$$\frac{\exists x\phi \qquad \boxed{\begin{array}{c}[x \Rightarrow x_0]\phi \\ \vdots \\ \chi\end{array}}^{\,x_0}_{\,[x\Rightarrow x_0]\phi}}{\chi}\,[\exists e]$$

If we know $\exists x\phi$, we know that there exist at least one object x for which ϕ holds. We call that element x_0, and assume $[x \Rightarrow x_0]\phi$ within a box in which we introdue x_0. Without assumptions on x_0, we prove a formula χ, in which x_0 does not appear. Since we have not made any assumptions on x_0, we can conclude from $\exists x\phi$ that χ holds.

Example 4. *We prove the following sequent:*

$$\forall x(P(x) \to Q(x)), \exists x P(x) \vdash \exists x Q(x)$$

1	$\forall x(P(x) \to Q(x))$	*premise*	
2	$\exists x P(x)$	*premise*	
3	$P(x_0)$	*assumption*	x_0
4	$P(x_0) \to Q(x_0)$	$\forall x$ *e 1*	
5	$Q(x_0)$	$\to$ *e 4,3*	
6	$\exists x Q(x)$	$\exists x$ *i 5*	
7	$\exists x Q(x)$	$\exists x$ *e 2,3–6*	

Note that $\exists x Q(x)$ within the box does not contain x_0, and therefore can be "exported" from the box.

Example 5. *We prove the following sequent:*

$$\forall x(Q(x) \to R(x)), \exists x(P(x) \land Q(x)) \vdash \exists x(P(x) \land R(x))$$

1	$\forall x(Q(x) \to R(x))$	*premise*	
2	$\exists x(P(x) \wedge Q(x))$	*premise*	
3	$P(x_0) \wedge Q(x_0)$	*assumption*	x_0
4	$Q(x_0) \to R(x_0)$	$\forall x$ *e 1*	
5	$Q(x_0)$	$\wedge e_2$ *3*	
6	$R(x_0)$	$\to$ *e 4,5*	
7	$P(x_0)$	$\wedge e_1$ *3*	
8	$P(x_0) \wedge R(x_0)$	$\wedge i$ *7, 6*	
9	$\exists x(P(x) \wedge R(x)$	$\exists x$ *i 8*	
10	$\exists x(P(x) \wedge R(x))$	$\exists x$ *e 2,3–9*	

Note that variables introduced by a box must be fresh! The following is not a proof, since the variable x_0 is introduced in nested boxes.

1	$\exists x P(x)$	premise	
2	$\forall x(P(x) \to Q(x))$	premise	
3			x_0
4	$P(x_0)$	assumption	x_0
5	$P(x_0) \to Q(x_0)$	$\forall x$ e 2	
6	$Q(x_0)$	$\to e$ 5,4	
7	$Q(x_0)$	$\exists x$ e 1, 4–6	
8	$\forall y Q(y)$	$\forall y$ i 3–7	

1.5 Equivalences

We write $\phi \dashv\vdash \psi$ iff $\phi \vdash \psi$ and also $\psi \vdash \phi$.

Lemma 1.

$$\begin{array}{rcl}
\neg\forall x\phi & \dashv\vdash & \exists x\neg\phi \\
\neg\exists x\phi & \dashv\vdash & \forall x\neg\phi \\
\forall x\forall y\phi & \dashv\vdash & \forall y\forall x\phi \\
\exists x\exists y\phi & \dashv\vdash & \exists y\exists x\phi \\
\forall x\phi \wedge \forall x\psi & \dashv\vdash & \forall x(\phi \wedge \psi) \\
\exists x\phi \vee \exists x\psi & \dashv\vdash & \exists x(\phi \vee \psi)
\end{array}$$

Proof. We shall prove the left-to-right directions of the first and fourth statement, and leave the remaining proofs to the reader. The proves are actually schemas; actual sequents and proofs are obtained by replacing ϕ and ψ with arbitrary formulas in a particular predicate logic.

- $\neg\forall x\phi \vdash \exists x\neg\phi$

1	$\neg\forall x\phi$	premise	
2	$\neg\exists x\neg\phi$	assumption	
3			x_0
4	$\neg[x \Rightarrow x_0]\phi$	assumption	
5	$\exists x\neg\phi$	$\exists x$ i 4	
6	$\perp$	$\neg e$ 5, 2	
7	$[x \Rightarrow x_0]\phi$	PBC 4–6	
8	$\forall x\phi$	$\forall x$ i 3–7	
9	$\perp$	$\neg e$ 8, 1	
10	$\exists x\neg\phi$	PBC 2–9	

- $\exists x\exists y\phi \vdash \exists y\exists x\phi$

  If x and y are the same variable, the left and write hand side are the same formula, and thus the sequent holds through a simple argument (for example conjunction introduction followed by elimination).

  Assume now that x and y are different variables.

1	$\exists x\exists y\phi$	premise	
2	$[x \Rightarrow x_0](\exists y\phi)$	assumption	x_0
3	$\exists y([x \Rightarrow x_0]\phi)$	def of subst (x, y different)	
4	$[y \Rightarrow y_0][x \Rightarrow x_0]\phi$	assumption	y_0
5	$[x \Rightarrow x_0][y \Rightarrow y_0]\phi$	def of subst (x, y, x_0, y_0 different)	
6	$\exists x[y \to y_0]\phi$	$\exists x$ i 5	
7	$\exists y\exists x\phi$	$\exists y$ i 6	
8	$\exists y\exists x\phi$	$\exists y$ e 3, 4–7	
9	$\exists y\exists x\phi$	$\exists x$ e 1, 2–8	

$\square$

Exercise 1. *Prove the remaining directions of the statements in Lemma 1.*

Lemma 2. *Assuming that x is not free in ψ, the following sequents hold:*

$$\begin{aligned}
\forall x\phi \wedge \psi &\dashv\vdash \forall x(\phi \wedge \psi)\\
\forall x\phi \vee \psi &\dashv\vdash \forall x(\phi \vee \psi)\\
\exists x\phi \wedge \psi &\dashv\vdash \exists x(\phi \wedge \psi)\\
\exists x\phi \vee \psi &\dashv\vdash \exists x(\phi \vee \psi)
\end{aligned}$$

Exercise 2. *Prove the statements of Lemma 2.*

2 Soundness, Completeness, Compactness

The following result justifies the use of natureal deduction in predicate logic.

Theorem 1 (Soundness and Completeness of Predicate Logic)**.**

$$\phi_1, \ldots, \phi_n \models \psi$$

iff

$$\phi_1, \ldots, \phi_n \vdash \psi$$

The theorem states that every valid sequent can be proven, and every sequent that can be proven is valid. This theorem was proven by Kurt Gödel in 1929 in his doctoral dissertation. A description of his proof, as well as the proofs of the following theorems, is beyond the scope of this chapter.

Theorem 2. *The decision problem of validity in predicate logic is undecidable: no program exists which, given any language in predicate logic and any formula ϕ in that language, decides whether $\models \phi$.*

Proof. (sketch)

- Establish that the Post Correspondence Problem (PCP) is undecidable
- Translate an arbitrary PCP, say C, to a formula ϕ.
- Establish that $\models \phi$ holds if and only if C has a solution.
- Conclude that validity of predicate logic formulas is undecidable.

□

Theorem 3. *Let Γ be a (possibly infinite) set of sentences of predicate logic. If all finite subsets of Γ are satisfiable, the Γ itself is satisfiable.*

Theorem 4 (Löwenheim-Skolem Theorem)**.** *Let ψ be a sentence of predicate logic such that for any natural number $n \geq 1$ there is a model of ψ with at least n elements. Then ψ has a model with infinitely many elements.*