03a—Induction

CS 3234: Logic and Formal Systems

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1. Inductive Definitions
   - What are Inductive Definitions?
   - Extremal Clause
   - Proofs by Induction
   - Defining Sets by Rules in Java

Inductive Definitions

- We will frequently define a set by a collection of rules that determine the elements of that set.
  Example: the set of valid sentences of a grammar
- What does it mean to define a set by a collection of rules?

Examples

- Numerals, in unary (base-1) notation.
  - *Zero* is a numeral;
  - if n is a numeral, then so is $Succ(n)$.
- Binary trees (w/o data at nodes):
  - *Empty* is a binary tree;
  - If l and r are binary trees, then so is $Node(l, r)$.

Examples (more formally)

- Numerals: The set *Num* is defined by the rules

$$\frac{}{\mathit{Zero}} \qquad \frac{n}{\mathit{Succ}(n)}$$

- Binary trees: The set *Tree* is defined by the rules

$$\frac{}{\mathit{Empty}} \qquad \frac{t_l \quad t_r}{\mathit{Node}(t_l, t_r)}$$

Defining a Set by Rules

- Given a collection of rules, what set does it define?
  - What is the set of numerals?
  - What is the set of trees?
- Do the rules pick out a unique set?

Defining a Set by Rules

- There can be many sets that satisfy a given collection of rules.
  - $\mathit{MyNum} = \{\mathit{Zero}, \mathit{Succ}(\mathit{Zero}), \ldots\}$
  - $\mathit{YourNum} = \mathit{MyNum} \cup \{\infty, \mathit{Succ}(\infty), \ldots\}$, where ∞ is an arbitrary symbol
- Both *MyNum* and *YourNum* satisfy the rules defining numerals (i.e., the rules are true for these sets). Really?

MyNum Satisfies the Rules

$$\frac{}{\mathit{Zero}} \qquad \frac{n}{\mathit{Succ}(n)}$$

$\mathit{MyNum} = \{\mathit{Zero}, \mathit{Succ}(\mathit{Zero}), \mathit{Succ}(\mathit{Succ}(\mathit{Zero})), \ldots\}$
Does *MyNum* satisfy the rules?

- $\mathit{Zero} \in \mathit{MyNum}$. $\surd$
- If $n \in \mathit{MyNum}$, then $\mathit{Succ}(n) \in \mathit{MyNum}$. $\surd$

YourNum Satisfies the Rules

$$\frac{}{\mathit{Zero}} \qquad \frac{n}{\mathit{Succ}(n)}$$

$\mathit{YourNum} =$
$\{\mathit{Zero}, \mathit{Succ}(\mathit{Zero}), \mathit{Succ}(\mathit{Succ}(\mathit{Zero})), \ldots\} \cup \{\infty, \mathit{Succ}(\infty), \ldots\}$
Does *YourNum* satisfy the rules?

- $\mathit{Zero} \in \mathit{YourNum}$. $\surd$
- If $n \in \mathit{YourNum}$, then $\mathit{Succ}(n) \in \mathit{YourNum}$. $\surd$

Defining Sets by Rules

- Both *MyNum* and *YourNum* satisfy all rules.
- It is not enough that a set satisfies all rules.
- Something more is needed: an *extremal* clause.
  - “and nothing else” or
  - “the least set that satisfies these rules”

Example 1: *Num*

Num is the least set that satisfies these rules:

- *Zero* is included
- If *n* is included, then *Succ*(*n*) is included.

Example 2: *Tree*

Tree is the least set that satisfies these rules:

- *Zero* is included
- If t_l and t_r are included, then $\mathit{Node}(t_l, t_r)$ is included.

Inductive Definitions

Question: What do we mean by “least”?
Answer: The smallest with respect to the subset ordering on sets.

- Contains no “junk”, only what is required by the rules.
- Since *YourNum* $\supsetneq$ *MyNum*, *YourNum* is ruled out by the extremal clause.
- *MyNum* is “ruled in” because it has no “junk”.

What's the Big Deal?

- Inductively defined sets "come with" an induction principle.
- Suppose I is inductively defined by rules R.
- To show that every $x \in I$ has property P, it is enough to show that P satisfies the rules of R.
- Sometimes called *structural induction* or *rule induction*.

Induction Principle

- To show that every $n \in \mathit{Num}$ has property P, it is enough to show:
  - *Zero* has property P.
  - if n has property P, then $\mathit{Succ}(n)$ has property P.
- This is just ordinary mathematical induction!

Induction Principle

- To show that every tree has property P, it is enough to show that
  - *Empty* has property P.
  - If l and r have property P, then so does $Node(l, r)$.
- We call this *structural induction on trees*.

Induction Principle

How can we justify this principle?

- Properties are sets. We are trying to show that $P \supseteq I$.
- Remember that I is (by definition) the smallest set satisfying the rules in R.
- Hence if P satisfies the rules of R, then $P \supseteq I$.
- This is why the extremal clause matters so much!

Example: Size of a Tree

- To show: Every tree has a size, defined as follows:

Definition of size

- The size of *Empty* is 1.
- If tree l has size h_l and tree r has size h_r, then the tree $Node(l, r)$ has size $1 + h_l + h_r$.

- Clearly, every tree has at most one size, but does it have a size at all?

Example: size

- It may seem obvious that every tree has a size, but notice that the justification relies on structural induction!
    - An “infinite tree” does not have a size!
    - But the extremal clause rules out the infinite tree!

Example: size

- We prove that every tree (as defined above) has a size (as defined above).
- Proceed by induction on the rules defining trees, showing that the property “has a size” satisfies the rules defining trees.
- Since the set of trees is the *least set* that satisfies the rules, the property “has a size” must be a superset of the set of trees!

Example: size

Definition of size

- The size of *Empty* is 1.
- If tree l has size h_l and tree r has size h_r, then the tree $Node(l, r)$ has size $1 + h_l + h_r$.

- Rule 1 of Def of Tree: *Empty* is included.
  Do all things that have a size fulfill this rule?
  Does *Empty* have a size? **yes**
- Rule 2 of Def of Tree: If l and r are included, then $Node(l, r)$ is included.
  Does all things that have a size fulfill this rule?
  If l and r have sizes, then $Node(l, r)$ has a size? **yes**

Example: size (summary)

- We have defined *Tree* as the least set satisfying:
  - *Zero* is included
  - If t_l and t_r are included, then $Node(t_l, t_r)$ is included.
- We have shown that the property "has a size" is a set satisfying
  - *Zero* is included
  - If t_l and t_r are included, then $Node(t_l, t_r)$ is included.
- Thus, the property "has a size" is a superset of *Tree*, meaning: Every *Tree* has a size.

Encoding Numerals in Java

```
interface Num {}
class Zero implements Num {}
class Succ implements Num {
   public Num pred;
   Succ(Num p) {pred = p;}
}
Num my_num = new Zero();
Num my_other_num =
   new Succ(new Succ(new Zero()));
```

Encoding Trees in Java

```
interface Tree {}
class Empty implements Tree {}
class Node implements Tree {
   public Tree left, right;
   Node(Tree l,Tree r) {
      left = l; right = r;}
}
Tree my_tree =
   new Node(new Empty(),
            new Node(new Node(new Empty(),
                              new Empty()),
                     new Empty()));
```

Constructors and Rules

- The constructors of the classes correspond to the rules in the inductive definition.
- Numerals
  - `new Zero()` is of type `Num`
  - if `n` is of type `Num`, then `new Succ(n)` is of type `Num`
- Trees
  - `new Empty()` is of type `Tree`
  - if `l` and `r` are of type `Tree`, then `new Node(l,r)` is of type `Tree`

Analogy with Java

- We assume an implicit extremal clause: no other classes implement the interface.
- The associated induction principle may be used to prove termination and correctness of functions.

Example: Size in Java

```
interface Tree {
   public int size();
}
class Empty implements Tree {
   public int size() {return 1;}
}
class Node implements Tree {
   public Tree left, right;
   Node(Tree l,Tree r) {left = l; right = r;}
   public int size() {
      return 1 + left.size() + right.size();
   }
}
```

Proving Termination of Java Program

Why does `size(t)` terminate for every tree `t`?

- For every `t` of type `Tree`, does there exist `h` such that `size(t)` returns `h`?
- Proof similar to above!

Summary

- An inductively defined set is the least set closed under a collection of rules.
- Rules have the form:
  "If $x_1 \in X$ and $\ldots$ and $x_n \in X$, then $x \in X$."
- Notation:

$$\frac{x_1 \quad \cdots \quad x_n}{x}$$

Summary

- Inductively defined sets admit proofs by rule induction.
- For each rule

$$\frac{x_1 \qquad \cdots \qquad x_n}{x}$$

  assume that $x_1 \in P$, ..., $x_n \in P$, and show that $x \in P$.
- Conclude that every element of the set is in P.