04b—Predicate Logic

CS 3234: Logic and Formal Systems

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More Declarative Sentences

- Propositional logic can easily handle simple declarative statements such as:

Example

Student Peter Lim enrolled in CS3234.

- Propositional logic can also handle combinations of such statements such as:

Example

Student Peter Lim enrolled in Tutorial 1, *and* student Julie Bradshaw is enrolled in Tutorial 2.

- *But:* How about statements with *"there exists..."* or *"every..."* or *"among..."*?

What is needed?

Example

Every student is younger than *some* instructor.

What is this statement about?

- Being a student
- Being an instructor
- Being younger than somebody else

These are *properties* of elements of a *set* of objects.

We express them in predicate logic using *predicates*.

Predicates

Example

Every student is younger than *some* instructor.

- $S(andy)$ could denote that Andy is a student.
- $I(paul)$ could denote that Paul is an instructor.
- $Y(andy, paul)$ could denote that Andy is younger than Paul.

The Need for Variables

Example

Every student is younger than *some* instructor.

We use the predicate S to denote student-hood.
How do we express *"every student"*?

We need *variables* that can stand for constant values, and a *quantifier* symbol that denotes *"every"*.

The Need for Variables

Example

Every student is younger than *some* instructor.

Using variables and quantifiers, we can write:

$$\forall x(S(x) \rightarrow (\exists y(I(y) \land Y(x, y)))).$$

Literally: For every x, if x is a student, then there is some y such that y is an instructor and x is younger than y.

Another Example

English

Not all birds can fly.

Predicates

$B(x)$: x is a bird

$F(x)$: x can fly

The sentence in predicate logic

$$\neg(\forall x(B(x) \to F(x)))$$

A Third Example

English

Every girl is younger than her mother.

Predicates

$G(x)$: x is a girl

$M(x, y)$: x is y's mother

$Y(x, y)$: x is younger than y

The sentence in predicate logic

$$\forall x \forall y (G(x) \land M(y, x) \to Y(x, y))$$

A “Mother” Function

The sentence in predicate logic

$$\forall x \forall y (G(x) \land M(y, x) \rightarrow Y(x, y))$$

Note that y is only introduced to denote the mother of x.

If everyone has exactly one mother, the predicate $M(y, x)$ is a function, when read from right to left.

We introduce a function symbol m that can be applied to variables and constants as in

$$\forall x (G(x) \rightarrow Y(x, m(x)))$$

A Drastic Example

English

Andy and Paul have the same maternal grandmother.

The sentence in predicate logic without functions

$$\forall x \forall y \forall u \forall v (M(x, y) \wedge M(y, andy) \wedge M(u, v) \wedge M(v, paul) \rightarrow x = u)$$

The same sentence in predicate logic with functions

$$m(m(andy)) = m(m(paul))$$

Outlook

Syntax: We formalize the language of predicate logic, including scoping and substitution.

Semantics: We describe models in which predicates, functions, and formulas have meaning.

Proof theory: We extend natural deduction from propositional to predicate logic (next week)

Further topics: Soundness/completeness, undecidability, incompleteness results, compactness results

Predicate Vocabulary

At any point in time, we want to describe the features of a particular "world", using predicates, functions, and constants. Thus, we introduce for this world:

- a set of predicate symbols $\mathcal{P}$
- a set of function symbols $\mathcal{F}$

Arity of Functions and Predicates

Every function symbol in $\mathcal{F}$ and predicate symbol in $\mathcal{P}$ comes with a fixed arity, denoting the number of arguments the symbol can take.

Special case: Nullary Functions

Function symbols with arity 0 are called *constants*.

Special case: Nullary Predicates

Predicate symbols with arity 0 denotes predicates that do not depend on any arguments. They correspond to propositional atoms.

Terms

$$t ::= \; x \mid c \mid f(t, \ldots, t)$$

where

- x ranges over a given set of variables $\mathcal{V}$,
- c ranges over nullary function symbols in $\mathcal{F}$, and
- f ranges over function symbols in $\mathcal{F}$ with arity $n > 0$.

Examples of Terms

If n is nullary, f is unary, and g is binary, then examples of terms are:

- $g(f(n), n)$
- $f(g(n, f(n)))$

More Examples of Terms

If $0, 1, 2$ are nullary (constants), s is unary, and $+, -$ and $*$ are binary, then

$$*(-(2, +(s(x), y)), x)$$

is a term.
Occasionally, we allow ourselves to use infix notation for function symbols as in

$$(2 - (s(x) + y)) * x$$

Formulas

$$\begin{aligned}\phi \quad ::= \quad & P(t, \ldots, t) \mid (\neg\phi) \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid \\ & (\phi \rightarrow \phi) \mid (\forall x \phi) \mid (\exists x \phi)\end{aligned}$$

where

- $P \in \mathcal{P}$ is a predicate symbol of arity $n \geq 0$,
- t are terms over $\mathcal{F}$ and $\mathcal{V}$, and
- x are variables in $\mathcal{V}$.

Conventions

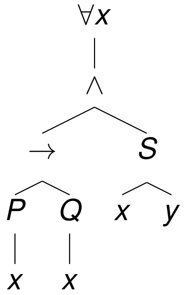
Just like for propositional logic, we introduce convenient conventions to reduce the number of parentheses:

- $\neg, \forall x$ and $\exists x$ bind most tightly;
- then $\wedge$ and $\vee$;
- then $\rightarrow$, which is right-associative.

Parse Trees

$$\forall x((P(x) \rightarrow Q(x)) \land S(x, y))$$

has parse tree



Another Example

Every son of my father is my brother.

Predicates

$S(x, y)$: x is a son of y

$B(x, y)$: x is a brother of y

Functions

m: constant for "me"

$f(x)$: father of x

The sentence in predicate logic

$$\forall x(S(x, f(m)) \rightarrow B(x, m))$$

Does this formula hold?

Equality as Predicate

Equality is a common predicate, usually used in infix notation.

$$= \in \mathcal{P}$$

Example

Instead of the formula

$$=(f(x), g(x))$$

we usually write the formula

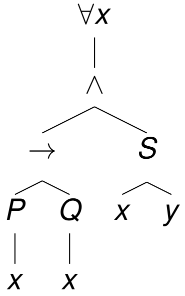
$$f(x) = g(x)$$

Free and Bound Variables

Consider the formula

$$\forall x((P(x) \to Q(x)) \land S(x, y))$$

What is the relationship between variable "binder" x and occurrences of x?

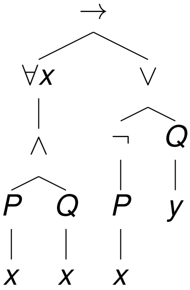


Free and Bound Variables

Consider the formula

$$(\forall x(P(x) \land Q(x))) \to (\neg P(x) \lor Q(y))$$

Which variable *occurrences* are free; which are bound?



Substitution

Variables are *place*holders. Re*plac*ing them by terms is called *substitution*.

Definition

Given a variable x, a term t and a formula ϕ, we define $[x \Rightarrow t]\phi$ to be the formula obtained by replacing each free occurrence of variable x in ϕ with t.

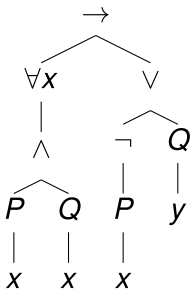
Example

$$[x \Rightarrow f(x,y)]((\forall x(P(x) \land Q(x))) \to (\neg P(x) \lor Q(y)))$$
$$= \forall x(P(x) \land Q(x))) \to (\neg P(f(x,y)) \lor Q(y))$$

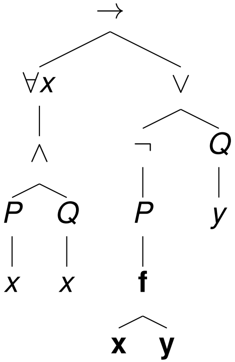
Example as Parse Tree

$$[x \Rightarrow f(x, y)]((\forall x(P(x) \wedge Q(x))) \rightarrow (\neg P(x) \vee Q(y)))$$

$$= (\forall x(P(x) \wedge Q(x))) \rightarrow (\neg P(f(x, y)) \vee Q(y))$$



Example as Parse Tree



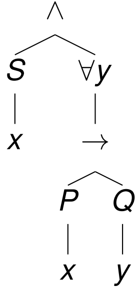
Capturing in $[x \Rightarrow t]\phi$

Problem

t contains variable y and x occurs under the scope of $\forall y$ in ϕ

Example

$$[x \Rightarrow f(y, y)](S(x) \land \forall y(P(x) \to Q(y)))$$



Avoiding Capturing

Definition

Given a term t, a variable x and a formula ϕ, we say that t is free for x in ϕ if no free x leaf in ϕ occurs in the scope of $\forall y$ or $\exists y$ for any variable y occurring in t.

Free-ness as precondition

In order to compute $[x \Rightarrow t]\phi$, we demand that t is free for x in ϕ.

What if not?

Rename the bound variable!

Example of Renaming

$$[x \Rightarrow f(y, y)](S(x) \land \forall y(P(x) \to Q(y)))$$

$$\Downarrow$$

$$[x \Rightarrow f(y, y)](S(x) \land \forall z(P(x) \to Q(z)))$$

$$\Downarrow$$

$$S(f(y, y)) \land \forall z(P(f(y, y)) \to Q(z))$$

1. Syntax of Predicate Logic
2. Predicate Logic as a Formal Language
3. Semantics of Predicate Logic
   - Models
   - Equality
   - Free Variables
   - Satisfaction and Entailment

Models

Definition

Let $\mathcal{F}$ contain function symbols and $\mathcal{P}$ contain predicate symbols. A model $\mathcal{M}$ for $(\mathcal{F}, \mathcal{P})$ consists of:

1. A non-empty set A, the *universe*;
2. for each nullary function symbol $f \in \mathcal{F}$ a concrete element $f^{\mathcal{M}} \in A$;
3. for each $f \in F$ with arity $n > 0$, a concrete function $f^{\mathcal{M}} : A^n \to A$;
4. for each $P \in \mathcal{P}$ with arity $n > 0$, a function $P^{\mathcal{M}} : U^n \to \{F, T\}$.
5. for each $P \in \mathcal{P}$ with arity $n = 0$, a value from $\{F, T\}$.

Example

Let $\mathcal{F} = \{e, \cdot\}$ and $\mathcal{P} = \{\leq\}$.
Let model $\mathcal{M}$ for $(\mathcal{F}, \mathcal{P})$ be defined as follows:

1. Let A be the set of binary strings over the alphabet $\{0, 1\}$;
2. let $e^{\mathcal{M}} = \epsilon$, the empty string;
3. let $\cdot^{\mathcal{M}}$ be defined such that $s_1 \cdot^{\mathcal{M}} s_2$ is the concatenation of the strings s_1 and s_2; and
4. let $\leq^{\mathcal{M}}$ be defined such that $s_1 \leq^{\mathcal{M}} s_2$ iff s_1 is a prefix of s_2.

Example (continued)

1. Let A be the set of binary strings over the alphabet $\{0, 1\}$;
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Some Elements of A

- 10001
- ϵ
- $1010 \cdot^{\mathcal{M}} 1100 = 10101100$
- $000 \cdot^{\mathcal{M}} \epsilon = 000$

Equality Revisited

Interpretation of equality

Usually, we require that the equality predicate $=$ is interpreted as same-ness.

Extensionality restriction

This means that allowable models are restricted to those in which $a =^{\mathcal{M}} b$ holds if and only if a and b are the same elements of the model's universe.

Example (continued)

1. Let A be the set of binary strings over the alphabet $\{0, 1\}$;
2. let $e^{\mathcal{M}} = \epsilon$, the empty string;
3. let $\cdot^{\mathcal{M}}$ be defined such that $s_1 \cdot^{\mathcal{M}} s_2$ is the concatenation of the strings s_1 and s_2; and
4. let $\leq^{\mathcal{M}}$ be defined such that $s_1 \leq^{\mathcal{M}} s_2$ iff s_1 is a prefix of s_2.

Equality in $\mathcal{M}$

- $000 =^{\mathcal{M}} 000$
- $001 \neq^{\mathcal{M}} 100$

Another Example

Let $\mathcal{F} = \{z, s\}$ and $\mathcal{P} = \{\leq\}$.
Let model $\mathcal{M}$ for $(\mathcal{F}, \mathcal{P})$ be defined as follows:

1. Let A be the set of natural numbers;
2. let $z^{\mathcal{M}} = 0$;
3. let $s^{\mathcal{M}}$ be defined such that $s(n) = n + 1$; and
4. let $\leq^{\mathcal{M}}$ be defined such that $n_1 \leq^{\mathcal{M}} n_2$ iff the natural number n_1 is less than or equal to n_2.

How To Handle Free Variables?

Idea

We can give meaning to formulas with free variables by providing an environment (lookup table) that assigns variables to elements of our universe:

$$l : \mathcal{V} \rightarrow A.$$

Environment extension

We define environment extension such that $l[x \mapsto a]$ is the environment that maps x to a and any other variable y to $l(y)$.

Satisfaction Relation

The model $\mathcal{M}$ satisfies ϕ with respect to environment l, written $\mathcal{M} \models_l \phi$:

- in case ϕ is of the form $P(t_1, t_2, \ldots, t_n)$, if $a_1, a_2, \ldots, a_n$ are the results of evaluating $t_1, t_2, \ldots, t_n$ with respect to l, and if $P^{\mathcal{M}}(a_1, a_2, \ldots, a_n) = T$;
- in case ϕ is of the form P, if $P^{\mathcal{M}} = T$;
- in case ϕ has the form $\forall x \psi$, if the $\mathcal{M} \models_{l[x \mapsto a]} \psi$ holds for all $a \in A$;
- in case ϕ has the form $\exists x \psi$, if the $\mathcal{M} \models_{l[x \mapsto a]} \psi$ holds for some $a \in A$;

Satisfaction Relation (continued)

- in case ϕ has the form $\neg\psi$, if $\mathcal{M} \models_I \psi$ does not hold;
- in case ϕ has the form $\psi_1 \lor \psi_2$, if $\mathcal{M} \models_I \psi_1$ holds or $\mathcal{M} \models_I \psi_2$ holds;
- in case ϕ has the form $\psi_1 \land \psi_2$, if $\mathcal{M} \models_I \psi_1$ holds and $\mathcal{M} \models_I \psi_2$ holds; and
- in case ϕ has the form $\psi_1 \rightarrow \psi_2$, if $\mathcal{M} \models_I \psi_2$ holds whenever $\mathcal{M} \models_I \psi_1$ holds.

Satisfaction of Closed Formulas

If a formula ϕ has no free variables, we call ϕ a *sentence*.
$\mathcal{M} \models_I \phi$ holds or does not hold regardless of the choice of I.
Thus we write $\mathcal{M} \models \phi$ or $\mathcal{M} \not\models \phi$.

Semantic Entailment and Satisfiability

Let Γ be a possibly infinite set of formulas in predicate logic and ψ a formula.

Entailment

$\Gamma \models \psi$ iff for all models $\mathcal{M}$ and environments l, whenever $\mathcal{M} \models_l \phi$ holds for all $\phi \in \Gamma$, then $\mathcal{M} \models_l \psi$.

Satisfiability of Formulas

ψ is satisfiable iff there is some model $\mathcal{M}$ and some environment l such that $\mathcal{M} \models_l \psi$ holds.

Satisfiability of Formula Sets

Γ is satisfiable iff there is some model $\mathcal{M}$ and some environment l such that $\mathcal{M} \models_l \phi$, for all $\phi \in \Gamma$.

Semantic Entailment and Satisfiability

Let Γ be a possibly infinite set of formulas in predicate logic and ψ a formula.

Validity

ψ is valid iff for all models $\mathcal{M}$ and environments l, we have $\mathcal{M} \models_l \psi$.

The Problem with Predicate Logic

Entailment ranges over models

Semantic entailment between sentences: $\phi_1, \phi_2, \ldots, \phi_n \models \psi$ requires that in *all* models that satisfy $\phi_1, \phi_2, \ldots, \phi_n$, the sentence ψ is satisfied.

How to effectively argue about all possible models?

Usually the number of models is infinite; it is very hard to argue on the semantic level in predicate logic.

Idea from propositional logic

Can we use natural deduction for showing entailment?

Admin

- Coq Homework 2: out on module homepage; due 10/9, 9:30pm
- Assignment 3: out soon; due 9/9, 11:00am
- Monday, Wednesday: Office hours
- Tuesday: Tutorials (Assignments 2 and 3)
- Wednesday: Labs (Quiz 1 solution, Coq Homework 2)
- Thursday: Lecture on