05—Predicate Logic II

CS 3234: Logic and Formal Systems

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1. Review: Syntax and Semantics
   - Predicates, Functions, Terms, Formulas
   - Models
   - Satisfaction and Entailment
2. Proof Theory
3. Equivalences and Properties

Predicates

Example

Every student is younger than *some* instructor.

- $S(andy)$ could denote that Andy is a student.
- $I(paul)$ could denote that Paul is an instructor.
- $Y(andy, paul)$ could denote that Andy is younger than Paul.

Example

English

Every girl is younger than her mother.

Predicates

$G(x)$: x is a girl

$M(x, y)$: x is y's mother

$Y(x, y)$: x is younger than y

The sentence in predicate logic

$$\forall x \forall y (G(x) \land M(y, x) \rightarrow Y(x, y))$$

A “Mother” Function

The sentence in predicate logic

$$\forall x \forall y (G(x) \land M(y, x) \rightarrow Y(x, y))$$

The sentence using a function

$$\forall x (G(x) \rightarrow Y(x, m(x)))$$

Predicate Vocabulary

At any point in time, we want to describe the features of a particular "world", using predicates, functions, and constants. Thus, we introduce for this world:

- a set of predicate symbols $\mathcal{P}$
- a set of function symbols $\mathcal{F}$

Arity of Functions and Predicates

Every function symbol in $\mathcal{F}$ and predicate symbol in $\mathcal{P}$ comes with a fixed arity, denoting the number of arguments the symbol can take.

Special case: Nullary Functions

Function symbols with arity 0 are called *constants*.

Special case: Nullary Predicates

Predicate symbols with arity 0 denotes predicates that do not depend on any arguments. They correspond to propositional atoms.

Terms

$$t ::= \; x \mid c \mid f(t, \ldots, t)$$

where

- x ranges over a given set of variables $\mathcal{V}$,
- c ranges over nullary function symbols in $\mathcal{F}$, and
- f ranges over function symbols in $\mathcal{F}$ with arity $n > 0$.

Examples of Terms

If n is nullary, f is unary, and g is binary, then examples of terms are:

- $g(f(n), n)$
- $f(g(n, f(n)))$

Formulas

$$\phi \quad ::= \quad P(t, \ldots, t) \mid (\neg\phi) \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi) \mid (\forall x \phi) \mid (\exists x \phi)$$

where

- $P \in \mathcal{P}$ is a predicate symbol of arity $n \geq 0$,
- t are terms over $\mathcal{F}$ and $\mathcal{V}$, and
- x are variables in $\mathcal{V}$.

Equality as Predicate

Equality is a common predicate, usually used in infix notation.

$$= \in \mathcal{P}$$

Example

Instead of the formula

$$= (f(x), g(x))$$

we usually write the formula

$$f(x) = g(x)$$

Models

Definition

Let $\mathcal{F}$ contain function symbols and $\mathcal{P}$ contain predicate symbols. A model $\mathcal{M}$ for $(\mathcal{F}, \mathcal{P})$ consists of:

1. A non-empty set A, the *universe*;
2. for each nullary function symbol $f \in \mathcal{F}$ a concrete element $f^{\mathcal{M}} \in A$;
3. for each $f \in F$ with arity $n > 0$, a concrete function $f^{\mathcal{M}} : A^n \to A$;
4. for each $P \in \mathcal{P}$ with arity $n > 0$, a function $P^{\mathcal{M}} : U^n \to \{F, T\}$.
5. for each $P \in \mathcal{P}$ with arity $n = 0$, a value from $\{F, T\}$.

Equality Revisited

Interpretation of equality

Usually, we require that the equality predicate $=$ is interpreted as same-ness.

Extensionality restriction

This means that allowable models are restricted to those in which $a =^{\mathcal{M}} b$ holds if and only if a and b are the same elements of the model's universe.

Satisfaction Relation

The model $\mathcal{M}$ satisfies ϕ with respect to environment I, written $\mathcal{M} \models_I \phi$:

- in case ϕ is of the form $P(t_1, t_2, \ldots, t_n)$, if $a_1, a_2, \ldots, a_n$ are the results of evaluating $t_1, t_2, \ldots, t_n$ with respect to I, and if $P^{\mathcal{M}}(a_1, a_2, \ldots, a_n) = T$;
- in case ϕ is of the form P, if $P^{\mathcal{M}} = T$;
- in case ϕ has the form $\forall x \psi$, if the $\mathcal{M} \models_{I[x \mapsto a]} \psi$ holds for all $a \in A$;
- in case ϕ has the form $\exists x \psi$, if the $\mathcal{M} \models_{I[x \mapsto a]} \psi$ holds for some $a \in A$;

Satisfaction Relation (continued)

- in case ϕ has the form $\neg\psi$, if $\mathcal{M} \models_I \psi$ does not hold;
- in case ϕ has the form $\psi_1 \vee \psi_2$, if $\mathcal{M} \models_I \psi_1$ holds or $\mathcal{M} \models_I \psi_2$ holds;
- in case ϕ has the form $\psi_1 \wedge \psi_2$, if $\mathcal{M} \models_I \psi_1$ holds and $\mathcal{M} \models_I \psi_2$ holds; and
- in case ϕ has the form $\psi_1 \rightarrow \psi_2$, if $\mathcal{M} \models_I \psi_1$ holds whenever $\mathcal{M} \models_I \psi_2$ holds.

Semantic Entailment and Satisfiability

Let Γ be a possibly infinite set of formulas in predicate logic and ψ a formula.

Entailment

$\Gamma \models \psi$ iff for all models $\mathcal{M}$ and environments l, whenever $\mathcal{M} \models_l \phi$ holds for all $\phi \in \Gamma$, then $\mathcal{M} \models_l \psi$.

Satisfiability of Formulas

ψ is satisfiable iff there is some model $\mathcal{M}$ and some environment l such that $\mathcal{M} \models_l \psi$ holds.

Satisfiability of Formula Sets

Γ is satisfiable iff there is some model $\mathcal{M}$ and some environment l such that $\mathcal{M} \models_l \phi$, for all $\phi \in \Gamma$.

Semantic Entailment and Satisfiability

Let Γ be a possibly infinite set of formulas in predicate logic and ψ a formula.

Validity

ψ is valid iff for all models $\mathcal{M}$ and environments l, we have $\mathcal{M} \models_l \psi$.

The Problem with Predicate Logic

Entailment ranges over models

Semantic entailment between sentences: $\phi_1, \phi_2, \ldots, \phi_n \models \psi$ requires that in *all* models that satisfy $\phi_1, \phi_2, \ldots, \phi_n$, the sentence ψ is satisfied.

How to effectively argue about all possible models?

Usually the number of models is infinite; it is very hard to argue on the semantic level in predicate logic.

Idea from propositional logic

Can we use natural deduction for showing entailment?

1. Review: Syntax and Semantics
2. **Proof Theory**
   - Equality
   - Universal Quantification
   - Existential Quantification
3. Equivalences and Properties

Natural Deduction for Predicate Logic

Relationship between propositional and predicate logic

If we consider propositions as nullary predicates, propositional logic is a sub-language of predicate logic.

Inheriting natural deduction

We can translate the rules for natural deduction in propositional logic directly to predicate logic.

Example

$$\frac{\phi \qquad \psi}{\phi \wedge \psi}[\wedge i]$$

Built-in Rules for Equality

$$\frac{}{t = t}[= i] \qquad \frac{t_i = t_2 \qquad [x \Rightarrow t_1]\phi}{[x \Rightarrow t_2]\phi}[= e]$$

Properties of Equality

We show:

$$f(x) = g(x) \vdash h(g(x)) = h(f(x))$$

using

$$\frac{}{t = t}[= i] \qquad \frac{t_1 = t_2 \qquad [x \Rightarrow t_1]\phi}{[x \Rightarrow t_2]\phi}[= e]$$

1	$f(x) = g(x)$	premise
2	$h(f(x)) = h(f(x))$	$= i$
3	$h(g(x)) = h(f(x))$	$= e$ 1,2

Elimination of Universal Quantification

$$\frac{\forall x \phi}{[x \Rightarrow t]\phi}[\forall x\ e]$$

Once you have proven $\forall x \phi$, you can replace x by any term t in ϕ, provided that t is free for x in ϕ.

Example

$$\frac{\forall x \phi}{[x \Rightarrow t]\phi} [\forall x \; e]$$

We prove: $S(g(john)), \forall x(S(x) \rightarrow \neg L(x)) \vdash \neg L(g(john))$

1	$S(g(john))$	premise
2	$\forall x(S(x) \rightarrow \neg L(x))$	premise
3	$S(g(john)) \rightarrow \neg L(g(john))$	$\forall x \; e$ 2
4	$\neg L(g(john))$	$\rightarrow e$ 3,1

Introduction of Universal Quantification

$$\dfrac{\boxed{\begin{array}{c}\vdots\\ [x \Rightarrow x_0]\phi\end{array}}^{\,x_0}}{\forall x \phi}\;[\forall x\ i]$$

If we manage to establish a formula ϕ about a fresh variable x_0, we can assume $\forall x \phi$.
The variable x_0 must be *fresh*; we cannot introduce the same variable twice in nested boxes.

Example

$$\forall x(P(x) \rightarrow Q(x)), \forall x P(x) \vdash \forall x Q(x) \text{ via } \frac{\boxed{\begin{array}{c} \vdots \\ [x \Rightarrow x_0]\phi \end{array}}^{\,x_0}}{\forall x \phi}$$

1	$\forall x(P(x) \rightarrow Q(x))$	premise	
2	$\forall x P(x)$	premise	
3	$P(x_0) \rightarrow Q(x_0)$	$\forall x\ e$ 1	x_0
4	$P(x_0)$	$\forall x\ e$ 2	
5	$Q(x_0)$	$\rightarrow e$ 3,4	
6	$\forall x Q(x)$	$\forall x\ i$ 3–5	

Introduction of Existential Quantification

$$\frac{[x \Rightarrow t]\phi}{\exists x \phi}[\exists x\ i]$$

In order to prove $\exists x \phi$, it suffices to find a term t as "witness", provided that t is free for x in ϕ.

Example

$$\forall x \phi \vdash \exists x \phi$$

Recall: Definition of Models

A model $\mathcal{M}$ for $(\mathcal{F}, \mathcal{P})$ consists of:

1. A *non-empty* set U, the *universe*;
2. ...

Remark

Compare this with Traditional Logic (Coq Quiz 1).

Because U must not be empty, we should be able to prove the sequent above.

Example (continued)

$$\forall x \phi \vdash \exists x \phi$$

1	$\forall x \phi$	premise
2	$[x \Rightarrow x]\phi$	$\forall x\ e\ 1$
3	$\exists x \phi$	$\exists x\ i\ 2$

Elimination of Existential Quantification

$$\exists x \phi \qquad \boxed{\begin{array}{c}[x \Rightarrow x_0]\phi \\ \vdots \\ \chi\end{array}}^{\,x_0}_{\,[x\Rightarrow x_0]\phi}$$

$$\overline{\qquad\qquad\qquad\qquad\qquad\qquad}\;[\exists e]$$

$$\chi$$

Making use of $\exists$

If we know $\exists x \phi$, we know that there exist at least one object x for which ϕ holds. We call that element x_0, and assume $[x \Rightarrow x_0]\phi$. Without assumptions on x_0, we prove χ (x_0 not in χ).

Example

$$\forall x(P(x) \rightarrow Q(x)), \exists x P(x) \vdash \exists x Q(x)$$

1	$\forall x(P(x) \rightarrow Q(x))$	premise	
2	$\exists x P(x)$	premise	
3	$P(x_0)$	assumption	x_0
4	$P(x_0) \rightarrow Q(x_0)$	$\forall x\ e$ 1	
5	$Q(x_0)$	$\rightarrow e$ 4,3	
6	$\exists x Q(x)$	$\exists x\ i$ 5	
7	$\exists x Q(x)$	$\exists x\ e$ 2,3–6	

Note that $\exists x Q(x)$ within the box does not contain x_0, and therefore can be “exported” from the box.

Another Example

1	$\forall x(Q(x) \rightarrow R(x))$	premise	
2	$\exists x(P(x) \wedge Q(x))$	premise	
3	$P(x_0) \wedge Q(x_0)$	assumption	x_0
4	$Q(x_0) \rightarrow R(x_0)$	$\forall x\ e$ 1	
5	$Q(x_0)$	$\wedge e_2$ 3	
6	$R(x_0)$	$\rightarrow e$ 4,5	
7	$P(x_0)$	$\wedge e_1$ 3	
8	$P(x_0) \wedge R(x_0)$	$\wedge i$ 7, 6	
9	$\exists x(P(x) \wedge R(x)$	$\exists x\ i$ 8	
10	$\exists x(P(x) \wedge R(x))$	$\exists x\ e$ 2,3–9	

Variables must be fresh! This is not a proof!

1	$\exists x P(x)$	premise	
2	$\forall x (P(x) \to Q(x))$	premise	
3			x_0
4	$P(x_0)$	assumption	x_0
5	$P(x_0) \to Q(x_0)$	$\forall x\ e$ 2	
6	$Q(x_0)$	$\to e$ 5,4	
7	$Q(x_0)$	$\exists x\ e$ 1, 4–6	
8	$\forall y Q(y)$	$\forall y\ i$ 3–7	

1. Review: Syntax and Semantics
2. Proof Theory
3. Equivalences and Properties
   - Quantifier Equivalences
   - Soundness and Completeness
   - Undecidability, Compactness

Equivalences

Two-way-provable

We write $\phi \dashv\vdash \psi$ iff $\phi \vdash \psi$ and also $\psi \vdash \phi$.

Some simple equivalences

$$
\begin{aligned}
\neg\forall x\phi &\dashv\vdash \exists x\neg\phi \\
\neg\exists x\phi &\dashv\vdash \forall x\neg\phi \\
\forall x\forall y\phi &\dashv\vdash \forall y\forall x\phi \\
\exists x\exists y\phi &\dashv\vdash \exists y\exists x\phi \\
\forall x\phi \land \forall x\psi &\dashv\vdash \forall x(\phi \land \psi) \\
\exists x\phi \lor \exists x\psi &\dashv\vdash \exists x(\phi \lor \psi)
\end{aligned}
$$

$\neg\forall x\phi \vdash \exists x\neg\phi$

1	$\neg\forall x\phi$	premise
2	$\neg\exists x\neg\phi$	assumption
3		x_0
4	$\neg[x \Rightarrow x_0]\phi$	assumption
5	$\exists x\neg\phi$	$\exists x$ *i* 4
6	$\bot$	$\neg e$ 5, 2
7	$[x \Rightarrow x_0]\phi$	PBC 4–6
8	$\forall x\phi$	$\forall x$ *i* 3–7
9	$\bot$	$\neg e$ 8, 1
10	$\exists x\neg\phi$	PBC 2–9

$\exists x \exists y \phi \vdash \exists y \exists x \phi$

Assume that x and y are different variables.

1	$\exists x \exists y \phi$	premise	
2	$[x \Rightarrow x_0](\exists y \phi)$	assumption	x_0
3	$\exists y([x \Rightarrow x_0]\phi$	def of subst (x, y different)	
4	$[y \Rightarrow y_0][x \Rightarrow x_0]\phi$	assumption	y_0
5	$[x \Rightarrow x_0][y \Rightarrow y_0]\phi$	def of subst (x, y, x_0, y_0 different)	
6	$\exists x[y \to y_0]\phi$	$\exists x$ *i* 5	
7	$\exists y \exists x \phi$	$\exists y$ *i* 6	
8	$\exists y \exists x \phi$	$\exists y$ *e* 3, 4–7	
9	$\exists y \exists x \phi$	$\exists x$ *e* 1, 2–8	

More Equivalences

Assume that *x* is not free in ψ

$$\begin{aligned}
\forall x \phi \land \psi &\dashv\vdash \forall x (\phi \land \psi) \\
\forall x \phi \lor \psi &\dashv\vdash \forall x (\phi \lor \psi) \\
\exists x \phi \land \psi &\dashv\vdash \exists x (\phi \land \psi) \\
\exists x \phi \lor \psi &\dashv\vdash \exists x (\phi \lor \psi)
\end{aligned}$$

Central Result of Natural Deduction

$$\phi_1, \ldots, \phi_n \models \psi$$

iff

$$\phi_1, \ldots, \phi_n \vdash \psi$$

proven by Kurt Gödel, in 1929 in his doctoral dissertation

Recall: Decidability

Decision problems

A *decision problem* is a question in some formal system with a yes-or-no answer.

Decidability

Decision problems for which there is an algorithm that returns "yes" whenever the answer to the problem is "yes", and that returns "no" whenever the answer to the problem is "no", are called *decidable*.

Decidability of satisfiability

The question, whether a given propositional formula is satisifiable, is decidable.

Undecidability of Predicate Logic

Theorem

The decision problem of validity in predicate logic is undecidable: no program exists which, given any language in predicate logic and any formula ϕ in that language, decides whether $\models \phi$.

Proof sketch

- Establish that the Post Correspondence Problem (PCP) is undecidable
- Translate an arbitrary PCP, say C, to a formula ϕ.
- Establish that $\models \phi$ holds if and only if C has a solution.
- Conclude that validity of predicate logic formulas is undecidable.

Compactness

Theorem

Let Γ be a (possibly infinite) set of sentences of predicate logic. If all finite subsets of Γ are satisfiable, the Γ itself is satisfiable.

Application of Compactness

Theorem (Löwenheim-Skolem Theorem)

Let ψ be a sentence of predicate logic such that for any natural number n ≥ 1 there is a model of ψ with at least n elements. Then ψ has a model with infinitely many elements.

Next Week

- Induction (formal)
- Midterm test