# 03b—Inductive Definitions

CS 5209: Foundation in Logic and AI

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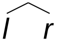
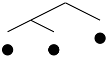
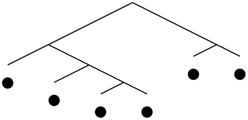
# Inductive definitions

- Often one wishes to define a set with a collection of rules that determine the elements of that set. Simple examples:
  - Binary trees
  - Natural numbers
- What does it mean to define a set by a collection of rules?

# Example 1: Binary trees (w/o data at nodes)

- ● is a binary tree;
- if *l* and *r* are binary trees, then so is 


Examples of binary trees:

- ●
- 
- 
- 

# Example 2: Natural numbers in unary (base-1) notation

- $Z$ is a natural;
- if $n$ is a natural, then so is $S(n)$.

We pronouce $Z$ as "zed" and "S" as successor. We can now define the natural numbers as follows:

$$\begin{array}{rcl} \textbf{zero} & \equiv & Z \\ \textbf{one} & \equiv & S(Z) \\ \textbf{two} & \equiv & S(S(Z)) \\ \ldots & & \end{array}$$

# It’s possible to view naturals as trees, too:

| | | | |
|---|---|---|---|
| **zero** | $\equiv$ | $Z$ | Z |
| **one** | $\equiv$ | $S(Z)$ |  |
| **two** | $\equiv$ | $S(S(Z))$ |  |

$\ldots$

# Examples (more formally)

- Binary trees: The set *Tree* is defined by the rules

$$\frac{}{\bullet} \qquad\qquad \frac{t_l \quad t_r}{\overset{\frown}{t_l \quad t_r}}$$

- Naturals: The set *Nat* is defined by the rules

$$\frac{}{Z} \qquad\qquad \frac{n}{S(n)}$$

# Given a collection of rules, what set does it define?

- What is the set of trees?
- What is the set of naturals?

Do the rules pick out a unique set?

# There can be many sets that satisfy a given collection of rules

- $\mathit{MyNum} = \{Z, S(Z), \ldots\}$
- $\mathit{YourNum} = \mathit{MyNum} \cup \{\infty, S(\infty), \ldots\}$, where $\infty$ is an arbitrary symbol.

Both *MyNum* and *YourNum* satisfy the rules defining numerals (i.e., the rules are true for these sets).

Really?

# *MyNum* Satisfies the Rules

$$\frac{}{Z} \qquad\qquad \frac{n}{S(n)}$$

$$MyNum = \{Z, Succ(Z), S(S(Z)), \ldots\}$$

Does *MyNum* satisfy the rules?

- $Z \in MyNum$. $\surd$
- If $n \in MyNum$, then $S(n) \in MyNum$. $\surd$

# *YourNum* Satisfies the Rules

$$\frac{}{Z} \qquad\qquad \frac{n}{S(n)}$$

$$YourNum = \{Z, S(Z), S(S(Z)), \ldots\} \cup \{\infty, S(\infty), \ldots\}$$

Does *YourNum* satisfy the rules?

- $Z \in YourNum$. $\surd$
- If $n \in YourNum$, then $S(n) \in YourNum$. $\surd$

# ... "And That's All!"

- Both *MyNum* and *YourNum* satisfy all rules.
- It is not enough that a set satisfies all rules.
- Something more is needed: an *extremal* clause.
  - "and nothing else"
  - "the least set that satisfies these rules"

# An inductively defined set is the **least set** for the given rules.

Example: $MyNum = \{Z, S(Z), S(S(Z)), \ldots\}$ is the least set that satisfies these rules:

- $Z \in Num$
- if $n \in Num$, then $S(n) \in Num$.

# What do we mean by “least”?

Answer: The smallest with respect to the subset ordering on sets.

- Contains no “junk”, only what is required by the rules.
- Since *YourNum* $\supsetneq$ *MyNum*, *YourNum* is ruled out by the extremal clause.
- *MyNum* is “ruled in” because it has no “junk”. That is, for any set *S* satisfying the rules, $S \supset MyNum$

# We almost always want to define sets with inductive definitions, and so have some simple notation to do so quickly:

$$S = \text{Constructor}_1(\ldots) \;\mid\; \text{Constructor}_2(\ldots) \;\mid\; \ldots$$

where $S$ can appear in the ... on the right hand side (along with other things). The $\text{Constructor}_i$ are the names of the different rules (sometimes text, sometimes symbols). This is called a *recursive definition*.

Examples:

- Binary trees: $\tau = \bullet \;\mid\; \widehat{\tau \quad \tau}$
- Naturals: $\mathbb{N} = Z \;\mid\; S(\mathbb{N})$

# There is a close connection between a recursive definition and a definition by rules:

- Binary trees: $\tau = \bullet \mid \widehat{\tau \quad \tau}$

$$\frac{}{\bullet} \qquad\qquad \frac{t_l \qquad t_r}{\widehat{t_l \quad t_r}}$$

- Naturals: $\mathbb{N} = Z \mid S(\mathbb{N})$

$$\frac{}{Z} \qquad\qquad \frac{n}{S(n)}$$

“recursive definition style” means that the extremal clause holds.

# What's the Big Deal?

Inductively defined sets "come with" an *induction principle*. Suppose $I$ is inductively defined by rules $R$.

- To show that every $x \in I$ has property $P$, it is enough to show that regardless of which rule is used to "build" x, $P$ holds; this is called *taking cases* or *inversion*.
- Sometimes, taking cases is not enough; in that case we can attempt a more complicated proof where we show that $P$ is preserved by each of the rules of $R$; this is called *structural induction* or *rule induction*.

# Example: Sign of a Natural

Consider the following definition:

- The natural $Z$ has sign **0**.
- For any natural $n$, the natural $S(n)$ has sign **1**.

Let $P$ be the following property: Every natural has sign **0** or **1**.

Does $P$ satisfy the rules $\dfrac{}{Z}$ $\dfrac{n}{S(n)}$ ?

# How to take cases

To show that every $n \in Nat$ has property $P$, it is enough to show:

- $Z$ has property $P$.
- For any $n$, $S(n)$ has property $P$.

Recall:

- The natural $Z$ has sign **0**.
- For any natural $n$, the natural $S(n)$ has sign **1.**

Let $P$ = "Every natural has sign **0** or **1**.". Does $P$ hold for all $\mathbb{N}$?

Proof. We take cases **on the structure of n** as follows:

- $Z$ has sign **0**, so $P$ holds for $Z$. $\surd$
- For any $n$, $S(n)$ has sign **1**, so $P$ holds for any $S(n)$. $\surd$

Thus, $P$ holds for all naturals.

# Example: Even and Odd Naturals

- The natural $Z$ has parity **0**.
- If $n$ is a natural with parity **0**, then $S(n)$ has parity **1**.
- If $n$ is a natural with parity **1**, then $S(n)$ has parity **0**.

Let $P$ be: Every natural has parity **0** or parity **1**.

Can we prove this by taking cases?

# Taking cases

We need to show $P$ = "Every natural has parity **0** or parity **1**.",

- $Z$ has property $P$.
- For any $n$, $S(n)$ has property $P$.

Where parity is defined by

- The natural $Z$ has parity **0**.
- If $n$ is a natural with parity **0**, then $S(n)$ has parity **1**.
- If $n$ is a natural with parity **1**, then $S(n)$ has parity **0**.

Proof. We take cases **on the structure of n** as follows:

- $Z$ has parity **0**, so $P$ holds for $Z$. $\surd$
- For any $n$, $S(n)$ has parity well... hmmm... it is unclear; it depends on the parity of $n$. ***X***

We are stuck! We need an extra fact about $n$'s parity...

# Induction hypothesis

This fact is called an *induction hypothesis*. To get such an induction hypothesis we do *induction*, which is a more powerful way to take cases. To show that every $n \in Num$ has property $P$, we must show that every rule preserves $P$; that is:

- $Z$ has property $P$.
- if $n$ has property $P$, then $S(n)$ has property $P$.

The new part is "if $n$ has property $P$, then ..."; this is the induction hypothesis.

Note that for the naturals, structural induction is just ordinary mathematical induction!

# Using induction to fix our proof

Every natural has parity **0** or parity **1**.

Proof. We take cases **on the structure of n** as follows:

- $Z$ has parity **0**, so $P$ holds for $Z$. $\surd$
- For any $n$, we can't determine the parity of $S(n)$ until we know something about the parity of $n$. ***X***

Proof. We **do induction on the structure of n** as follows:

- $Z$ has parity **0**, so $P$ holds for $Z$. $\surd$
- Given an $n$ such that $P$ holds on $n$, show that $P$ holds on $S(n)$. Since $P$ holds on $n$, the parity of $n$ is **0** or **1**. If the parity of $n$ is **0**, then the parity of $S(n)$ is **1**. If the parity of $n$ is **1**, then the parity of $S(n)$ is **0**. In either case, the parity of $S(n)$ is **0** or **1**, so if $P$ holds on $n$ then $P$ holds on $S(n)$. $\surd$

Thus, $P$ holds for an natural $n$.

# Extending case analysis and structural induction to trees

Case analysis: to show that every tree has property $P$, prove that

- $\bullet$ has property $P$.
- for all $\tau_1$ and $\tau_2$, $\overset{\wedge}{\tau_1 \quad \tau_2}$ has property $P$.

Structural induction: to show that every tree has property $P$, prove

- $\bullet$ has property $P$.
- if $\tau_1$ and $\tau_2$ have property $P$, then $\overset{\wedge}{\tau_1 \quad \tau_2}$ has property $P$.

Note that we do not require that $\tau_1$ and $\tau_2$ be the same height!

# How can we justify case analysis and induction?

Let $I$ be a set inductively defined by rules $R$.

- Case analysis is really a lightweight “special case” of structural induction where we do not use the induction hypothesis. If structural induction is sound, then case analysis will be as well.
- One way to think of a property $P$ is that it is exactly the set of items that have property $P$. We would like to show that if you are in the set $I$ then you have property $P$, that is, $P \supseteq I$.
- Remember that $I$ is (by definition) the smallest set satisfying the rules in $R$.
- Hence if $P$ satisfies (is preserved by) the rules of $R$, then $P \supseteq I$.
- This is why the extremal clause matters so much!

# Example: Height of a Tree

- To show: Every tree has a height, defined as follows:
  - The height of $\bullet$ is 0.
  - If the tree $l$ has height $h_l$ and the tree $r$ has height $h_r$, then the tree $\widehat{l \quad r}$ has height $1 + max(h_l, h_r)$.
- Clearly, every tree has at most one height, but does it have any height at all?
- It may seem obvious that every tree has a height, but notice that the justification relies on structural induction!
  - An “infinite tree” does not have a height!
  - But the extremal clause rules out the infinite tree!

# Example: height

- Formally, we prove that for every tree *t*, there exists a number *h* satisfying the specification of height.
- Proceed by induction **on the structure of trees**, showing that the property "there exists a height *h* for *t*" satisfies (is preserved by) these rules.

# Example: height

- Rule 1: $\bullet$ is a tree.
  Does there exist $h$ such that $h$ is the height of *Empty*?
  Yes! Take h=0.
- Rule 2: $\widehat{l \quad r}$ is a tree if $l$ and $r$ are trees.
  Suppose that there exists $h_l$ and $h_r$, the heights of $l$ and $r$, respectively (*the induction hypothesis*).
  Does there exist $h$ such that $h$ is the height of $Node(l, r)$?
  Yes! Take $h = 1 + max(h_l, h_r)$.

Thus, we have proved that all trees have a height.