03b—Inductive Definitions

CS 5209: Foundation in Logic and AI

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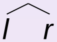
Inductive definitions

- Often one wishes to define a set with a collection of rules that determine the elements of that set. Simple examples:
  - Binary trees
  - Natural numbers
- What does it mean to define a set by a collection of rules?

Example 1: Binary trees (w/o data at nodes)

- • is a binary tree;
- if l and r are binary trees, then so is



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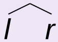
- if l and r are binary trees, then so is 

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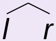
- if l and r are binary trees, then so is 

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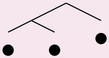
- 

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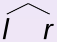
- if l and r are binary trees, then so is 

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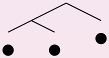
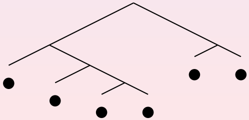
- 
- 

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\ldots & &
\end{array}
$$

It's possible to view naturals as trees, too:

zero	$\equiv$	Z	Z
one	$\equiv$	$S(Z)$	S | Z
two	$\equiv$	$S(S(Z))$	S | S | Z

$\ldots$

Examples (more formally)

- Binary trees: The set *Tree* is defined by the rules

$$\frac{}{\bullet} \qquad\qquad \frac{t_l \quad t_r}{\widehat{t_l \quad t_r}}$$

- Naturals: The set *Nat* is defined by the rules

$$\frac{}{Z} \qquad\qquad \frac{n}{S(n)}$$

Given a collection of rules, what set does it define?

- What is the set of trees?
- What is the set of naturals?

Do the rules pick out a unique set?

There can be many sets that satisfy a given collection of rules

- $\mathit{MyNum} = \{Z, S(Z), \ldots\}$
- $\mathit{YourNum} = \mathit{MyNum} \cup \{\infty, S(\infty), \ldots\}$, where ∞ is an arbitrary symbol.

Both *MyNum* and *YourNum* satisfy the rules defining numerals (i.e., the rules are true for these sets).

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Really?

MyNum Satisfies the Rules

$$\frac{}{Z} \qquad \frac{n}{S(n)}$$

$$MyNum = \{Z, Succ(Z), S(S(Z)), \ldots\}$$

Does *MyNum* satisfy the rules?

- $Z \in MyNum$. $\surd$
- If $n \in MyNum$, then $S(n) \in MyNum$. $\surd$

YourNum Satisfies the Rules

$$\frac{}{Z} \qquad \frac{n}{S(n)}$$

$$YourNum = \{Z, S(Z), S(S(Z)), \ldots\} \cup \{\infty, S(\infty), \ldots\}$$

Does *YourNum* satisfy the rules?

- $Z \in YourNum$. $\surd$
- If $n \in YourNum$, then $S(n) \in YourNum$. $\surd$

... "And That's All!"

- Both *MyNum* and *YourNum* satisfy all rules.
- It is not enough that a set satisfies all rules.
- Something more is needed: an *extremal* clause.
    - "and nothing else"
    - "the least set that satisfies these rules"

An inductively defined set is the **least set** for the given rules.

Example: $MyNum = \{Z, S(Z), S(S(Z)), \ldots\}$ is the least set that satisfies these rules:

- $Z \in Num$
- if $n \in Num$, then $S(n) \in Num$.

What do we mean by "least"?

Answer: The smallest with respect to the subset ordering on sets.

- Contains no "junk", only what is required by the rules.
- Since *YourNum* $\supsetneq$ *MyNum*, *YourNum* is ruled out by the extremal clause.
- *MyNum* is "ruled in" because it has no "junk". That is, for any set S satisfying the rules, $S \supset MyNum$

We almost always want to define sets with inductive definitions, and so have some simple notation to do so quickly:

$$S = \text{Constructor}_1(\ldots) \mid \text{Constructor}_2(\ldots) \mid \ldots$$

where S can appear in the $\ldots$ on the right hand side (along with other things). The Constructor_i are the names of the different rules (sometimes text, sometimes symbols). This is called a *recursive definition*.

Examples:

- Binary trees: $\tau = \bullet \mid \overset{\frown}{\tau \quad \tau}$
- Naturals: $\mathbb{N} = Z \mid S(\mathbb{N})$

There is a close connection between a recursive definition and a definition by rules:

- Binary trees: $\tau = \bullet \mid \widehat{\tau \quad \tau}$

$$\frac{}{\bullet} \qquad\qquad \frac{t_l \quad t_r}{\widehat{t_l \quad t_r}}$$

- Naturals: $\mathbb{N} = Z \mid S(\mathbb{N})$

$$\frac{}{Z} \qquad\qquad \frac{n}{S(n)}$$

"recursive definition style" means that the extremal clause holds.

What's the Big Deal?

Inductively defined sets "come with" an *induction principle*.
Suppose I is inductively defined by rules R.

- To show that every $x \in I$ has property P, it is enough to show that regardless of which rule is used to "build" x, P holds; this is called *taking cases* or *inversion*.
- Sometimes, taking cases is not enough; in that case we can attempt a more complicated proof where we show that P is preserved by each of the rules of R; this is called *structural induction* or *rule induction*.

Example: Sign of a Natural

Consider the following definition:

- The natural Z has sign **0**.
- For any natural n, the natural $S(n)$ has sign **1**.

Let P be the following property: Every natural has sign **0** or **1**.

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Does P satisfy the rules $\dfrac{}{Z}$ $\dfrac{n}{S(n)}$?

How to take cases

To show that every $n \in Nat$ has property P, it is enough to show:

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- For any n, $S(n)$ has sign **1**, so P holds for any $S(n)$. $\surd$

Thus, P holds for all naturals.

Example: Even and Odd Naturals

- The natural Z has parity **0**.
- If n is a natural with parity **0**, then $S(n)$ has parity **1**.
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Can we prove this by taking cases?

Taking cases

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We are stuck! We need an extra fact about n's parity…

Induction hypothesis

This fact is called an *induction hypothesis*. To get such an induction hypothesis we do *induction*, which is a more powerful way to take cases. To show that every $n \in Num$ has property P, we must show that every rule preserves P; that is:

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Note that for the naturals, structural induction is just ordinary mathematical induction!

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Every natural has parity **0** or parity **1**.

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Proof. We **do induction on the structure of n** as follows:

- Z has parity **0**, so P holds for Z. $\surd$
- Given an n such that P holds on n, show that P holds on $S(n)$. Since P holds on n, the parity of n is **0** or **1**. If the parity of n is **0**, then the parity of $S(n)$ is **1**. If the parity of n is **1**, then the parity of $S(n)$ is **0**. In either case, the parity of $S(n)$ is **0** or **1**, so if P holds on n then P holds on $S(n)$. $\surd$

Thus, P holds for an natural n.

Extending case analysis and structural induction to trees

Case analysis: to show that every tree has property P, prove that

- $\bullet$ has property P.
- for all τ_1 and τ_2, $\overset{\wedge}{\tau_1 \quad \tau_2}$ has property P.

Structural induction: to show that every tree has property P, prove

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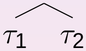
- for all τ_1 and τ_2, 

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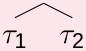
- if τ_1 and τ_2 have property P, then 

has property P.

Note that we do not require that τ_1 and τ_2 be the same height!

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- Hence if P satisfies (is preserved by) the rules of R, then $P \supseteq I$.
- This is why the extremal clause matters so much!

Example: Height of a Tree

- To show: Every tree has a height, defined as follows:
  - The height of $\bullet$ is 0.
  - If the tree l has height h_l and the tree r has height h_r, then the tree $\overset{\frown}{l \quad r}$ has height $1 + max(h_l, h_r)$.
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- Clearly, every tree has at most one height, but does it have any height at all?
- It may seem obvious that every tree has a height, but notice that the justification relies on structural induction!
  - An "infinite tree" does not have a height!
  - But the extremal clause rules out the infinite tree!

Example: height

- Formally, we prove that for every tree t, there exists a number h satisfying the specification of height.
- Proceed by induction **on the structure of trees**, showing that the property “there exists a height h for t” satisfies (is preserved by) these rules.

Example: height

- Rule 1: $\bullet$ is a tree.
  Does there exist h such that h is the height of *Empty*?
  Yes! Take h=0.
- Rule 2: $\overset{\frown}{l \quad r}$ is a tree if l and r are trees.
  Suppose that there exists h_l and h_r, the heights of l and r, respectively (*the induction hypothesis*).
  Does there exist h such that h is the height of $Node(l, r)$?
  Yes! Take $h = 1 + max(h_l, h_r)$.

Thus, we have proved that all trees have a height.