10—Modal Logic I

CS 5209: Foundation in Logic and AI

Martin Henz and Aquinas Hobor

March 25, 2010

Generated on Thursday 25th March, 2010, 16:24

1 Motivation

2 Basic Modal Logic

3 Logic Engineering

1 Motivation

2 Basic Modal Logic

3 Logic Engineering

Necessity

- You are crime investigator and consider different suspects.
  - Maybe the cook did it with a knife?
  - Maybe the maid did it with a pistol?
- But: "The victim Ms Smith made the call *before* she was killed." is *necessarily* true.
- "Necessarily" means in all possible scenarios (worlds) under consideration.

Notions of Truth

- Often, it is not enough to distinguish between "true" and "false".
- We need to consider *modalities* if truth, such as:
    - necessity ("in all possible scenarios")
    - morality/law ("in acceptable/legal scenarios")
    - time ("forever in the future")
- Modal logic constructs a framework using which modalities can be formalized and reasoning methods can be established.

1 Motivation

2 Basic Modal Logic
  ○ Syntax
  ○ Semantics
  ○ Equivalences

3 Logic Engineering

Syntax of Basic Modal Logic

$$
\begin{aligned}
\phi \quad ::= \quad & \top \mid \bot \mid \boldsymbol{p} \mid (\neg\phi) \mid (\phi \wedge \phi) \\
& \mid (\phi \vee \phi) \mid (\phi \to \phi) \\
& \mid (\phi \leftrightarrow \phi) \\
& \mid (\Box\phi) \mid (\Diamond\phi)
\end{aligned}
$$

Pronunciation and Examples

Pronunciation

If we want to keep the meaning open, we simply say "box" and "diamond".

If we want to appeal to our intuition, we may say "necessarily" and "possibly" (or "forever in the future" and "sometime in the future")

Examples

$$(p \land \Diamond(p \rightarrow \Box \neg r))$$

$$\Box((\Diamond q \land \neg r) \rightarrow \Box p)$$

Kripke Models

Definition

A model $\mathcal{M}$ of basic modal logic is specified by three things:

1. A set W, whose elements are called *worlds*;

2. A relation R on W, meaning $R \subseteq W \times W$, called the accessibility relation;

3. A function $L : W \to \mathcal{P}(\texttt{Atoms})$, called the labeling function.

Who is Kripke?

How do I know I am not dreaming? Kripke asked himself this question in 1952, at the age of 12. His father told him about the philosopher Descartes.

Modal logic at 17 Kripke's self-studies in philosophy and logic led him to prove a fundamental completeness theorem on modal logic at the age of 17.

Bachelor in Mathematics from Harvard is his only non-honorary degree

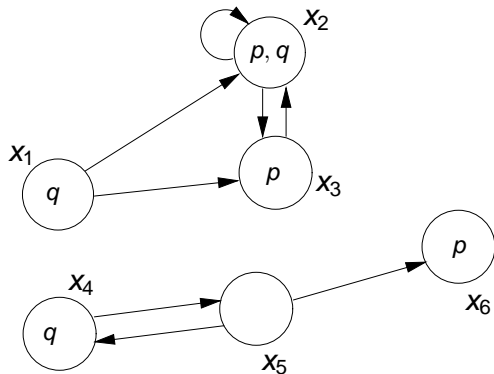At Princeton Kripke taught philosophy from 1977 onwards.

Contributions include modal logic, naming, belief, truth, the meaning of "I"

Example

$W = \{x_1, x_2, x_3, x_4, x_5, x_6\}$

$R = \{(x_1, x_2), (x_1, x_3), (x_2, x_2), (x_2, x_3), (x_3, x_2), (x_4, x_5), (x_5, x_4), (x_5, x_6)\}$

$L = \{(x_1, \{q\}), (x_2, \{p, q\}), (x_3, \{p\}), (x_4, \{q\}), (x_5, \{\}), (x_6, \{p\})\}$

When is a formula true in a possible world?

Definition

Let $\mathcal{M} = (W, R, L)$, $x \in W$, and ϕ a formula in basic modal logic. We define $x \Vdash \phi$ via structural induction:

- $x \Vdash \top$
- $x \nVdash \bot$
- $x \Vdash p$ iff $p \in L(x)$
- $x \Vdash \neg\phi$ iff $x \nVdash \phi$
- $x \Vdash \phi \wedge \psi$ iff $x \Vdash \phi$ and $x \Vdash \psi$
- $x \Vdash \phi \vee \psi$ iff $x \Vdash \phi$ or $x \Vdash \psi$
- ...

When is a formula true in a possible world?

Definition (continued)

Let $\mathcal{M} = (W, R, L)$, $x \in W$, and ϕ a formula in basic modal logic. We define $x \Vdash \phi$ via structural induction:

- ...

- $x \Vdash \phi \to \psi$ iff $x \Vdash \psi$, whenever $x \Vdash \phi$

- $x \Vdash \phi \leftrightarrow \psi$ iff ($x \Vdash \phi$ iff $x \Vdash \psi$)

- $x \Vdash \Box\phi$ iff for each $y \in W$ with $R(x, y)$, we have $y \Vdash \phi$

- $x \Vdash \Diamond\phi$ iff there is a $y \in W$ such that $R(x, y)$ and $y \Vdash \phi$.

Example
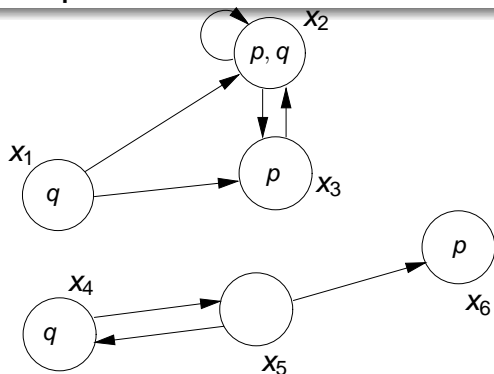
- $x_1 \Vdash q$
- $x_1 \Vdash \Diamond q$, $x_1 \not\Vdash \Box q$
- $x_5 \not\Vdash \Box p$, $x_5 \not\Vdash \Box q$, $x_5 \not\Vdash \Box p \vee \Box q$, $x_5 \Vdash \Box(p \vee q)$
- $x_6 \Vdash \Box \phi$ holds for all ϕ, but $x_6 \not\Vdash \Diamond \phi$

Formula Schemes

Example

We said $x_6 \Vdash \Box\phi$ holds for all ϕ, but $x_6 \not\Vdash \Diamond\phi$

Notation

Greek letters denote formulas, and are not propositional atoms.

Formula schemes

Terms where Greek letters appear instead of propositional atoms are called *formula schemes*.

Motivation   **Syntax**
Basic Modal Logic   **Semantics**
Logic Engineering   **Equivalences**

Entailment and Equivalence

Definition

A set of formulas Γ entails a formula ψ of basic modal logic if, in any world x of any model $\mathcal{M} = (W, R, L)$, whe have $x \Vdash \psi$ whenever $x \Vdash \phi$ for all $\phi \in$ Γ. We say Γ entails ψ and write $\Gamma \models \psi$.

Equivalence

We write $\phi \equiv \psi$ if $\phi \models \psi$ and $\psi \models \phi$.

Motivation   Syntax
Basic Modal Logic   Semantics
Logic Engineering   **Equivalences**

Some Equivalences

- De Morgan rules: $\neg\Box\phi \equiv \Diamond\neg\phi$, $\neg\Diamond\phi \equiv \Box\neg\phi$.
- Distributivity of $\Box$ over $\wedge$:

$$\Box(\phi \wedge \psi) \equiv \Box\phi \wedge \Box\psi$$

- Distributivity of $\Diamond$ over $\vee$:

$$\Diamond(\phi \vee \psi) \equiv \Diamond\phi \vee \Diamond\psi$$

- $\Box\top \equiv \top$, $\Diamond\bot \equiv \bot$

Motivation   Syntax
Basic Modal Logic   Semantics
Logic Engineering   **Equivalences**

Validity

Definition

A formula ϕ is valid if it is true in every world of every model, i.e. iff $\models \phi$ holds.

Examples of Valid Formulas

- All valid formulas of propositional logic
- $\neg\Box\phi \leftrightarrow \Diamond\neg\phi$
- $\Box(\phi \wedge \psi) \leftrightarrow \Box\phi \wedge \Box\psi$
- $\Diamond(\phi \vee \psi) \leftrightarrow \Diamond\phi \vee \Diamond\psi$
- Formula K: $\Box(\phi \rightarrow \psi) \wedge \Box\phi \rightarrow \Box\psi$.

Motivation
Basic Modal Logic
Logic Engineering

Valid Formulas wrt Modalities
Properties of R
Correspondence Theory
Preview: Some Modal Logics

1 Motivation

2 Basic Modal Logic

3 Logic Engineering
- Valid Formulas wrt Modalities
- Properties of R
- Correspondence Theory
- Preview: Some Modal Logics

Motivation
Basic Modal Logic
Logic Engineering

Valid Formulas wrt Modalities
Properties of R
Correspondence Theory
Preview: Some Modal Logics

A Range of Modalities

In a particular context $\Box\phi$ could mean:

- It is necessarily true that ϕ
- It will always be true that ϕ
- It ought to be that ϕ
- Agent Q believes that ϕ
- Agent Q knows that ϕ
- After any execution of program P, ϕ holds.

Since $\Diamond\phi \equiv \neg\Box\neg\phi$, we can infer the meaning of $\Diamond$ in each context.

Motivation
Basic Modal Logic
Logic Engineering

Valid Formulas wrt Modalities
Properties of R
Correspondence Theory
Preview: Some Modal Logics

A Range of Modalities

From the meaning of $\Box\phi$, we can conclude the meaning of $\Diamond\phi$, since $\Diamond\phi \equiv \neg\Box\neg\phi$:

$\Box\phi$	$\Diamond\phi$
It is necessarily true that ϕ	It is possibly true that ϕ
It will always be true that ϕ	Sometime in the future ϕ
It ought to be that ϕ	It is permitted to be that ϕ
Agent Q believes that ϕ	ϕ is consistent with Q's beliefs
Agent Q knows that ϕ	For all Q knows, ϕ
After any run of P, ϕ holds.	After some run of P, ϕ holds

Motivation
Basic Modal Logic
Logic Engineering

Valid Formulas wrt Modalities
Properties of R
Correspondence Theory
Preview: Some Modal Logics

Formula Schemes that hold wrt some Modalities

$\Box\phi$	$\Box\phi \to \phi$	$\Box\phi \to \Box\Box\phi$	$\Diamond\phi \to \Box\Diamond\phi$	$\Box\top$	$\Box\phi \to \Diamond\phi$	$\Diamond\phi \vee \Box\neg\phi$	$\Box(\phi \to \psi) \wedge \Box\phi \to \Box\psi$	$\Box\phi \wedge \Box\psi \to \Box(\phi \wedge \psi)$
It is necessary that ϕ	√	√	√	√	√	×	√	×
It will always be that ϕ	×	√	×	×	×	×	√	×
It ought to be that ϕ	×	×	×	√	√	×	√	×
Agent Q believes that ϕ	×	√	√	√	√	×	√	×
Agent Q knows that ϕ	√	√	√	√	√	×	√	×
After running P, ϕ	×	×	×	×	×	×	√	×

Motivation
Basic Modal Logic
Logic Engineering

Valid Formulas wrt Modalities
Properties of R
Correspondence Theory
Preview: Some Modal Logics

Modalities lead to Interpretations of R

$\Box\phi$	$R(x, y)$
It is necessarily true that ϕ	y is possible world according to info at x
It will always be true that ϕ	y is a future world of x
It ought to be that ϕ	y is an acceptable world according to the information at x
Agent Q believes that ϕ	y could be the actual world according to Q's beliefs at x
Agent Q knows that ϕ	y could be the actual world according to Q's knowledge at x
After any execution of P, ϕ holds	y is a possible resulting state after execution of P at x

Motivation
Basic Modal Logic
Logic Engineering

Valid Formulas wrt Modalities
Properties of R
Correspondence Theory
Preview: Some Modal Logics

Possible Properties of R

- reflexive: for every $w \in W$, we have $R(x, x)$.
- symmetric: for every $x, y \in W$, we have $R(x, y)$ implies $R(y, x)$.
- serial: for every x there is a y such that $R(x, y)$.
- transitive: for every $x, y, z \in W$, we have $R(x, y)$ and $R(y, z)$ imply $R(x, z)$.
- Euclidean: for every $x, y, z \in W$ with $R(x, y)$ and $R(x, z)$, we have $R(y, z)$.
- functional: for each x there is a unique y such that $R(x, y)$.
- linear: for every $x, y, z \in W$ with $R(x, y)$ and $R(x, z)$, we have $R(y, z)$ or $y = z$ or $R(z, y)$.
- total: for every $x, y \in W$, we have $R(x, y)$ and $R(y, x)$.
- equivalence: reflexive, symmetric and transitive.

Motivation
Basic Modal Logic
Logic Engineering

Valid Formulas wrt Modalities
Properties of R
Correspondence Theory
Preview: Some Modal Logics

Example

Consider the modality in which $\Box\phi$ means
"it ought to be that ϕ".

- Should R be reflexive?
- Should R be serial?

Motivation
Basic Modal Logic
Logic Engineering

Valid Formulas wrt Modalities
Properties of R
Correspondence Theory
Preview: Some Modal Logics

Necessarily true and Reflexivity

Guess

R is reflexive if and only if $\Box\phi \to \phi$ is valid.

Motivation
Basic Modal Logic
Logic Engineering

Valid Formulas wrt Modalities
Properties of R
Correspondence Theory
Preview: Some Modal Logics

Motivation

- We would like to establish that some formulas hold whenever R has a particular property.
- Ignore L, and only consider the (W, R) part of a model, called *frame*.
- Establish formula schemes based on properties of frames.

Motivation
Basic Modal Logic
Logic Engineering

Valid Formulas wrt Modalities
Properties of R
Correspondence Theory
Preview: Some Modal Logics

Reflexivity and Transitivity

Theorem 1

Let $\mathcal{F} = (W, R)$ be a frame. The following statements are equivalent:

- R is reflexive;
- $\mathcal{F}$ satisfies $\Box\phi \rightarrow \phi$;
- $\mathcal{F}$ satisfies $\Box p \rightarrow p$ for any atom p

Theorem 2

The following statements are equivalent:

- R is transitive;
- $\mathcal{F}$ satisfies $\Box\phi \rightarrow \Box\Box\phi$;
- $\mathcal{F}$ satisfies $\Box p \rightarrow \Box\Box p$ for any atom p

Motivation
Basic Modal Logic
Logic Engineering

Valid Formulas wrt Modalities
Properties of R
Correspondence Theory
Preview: Some Modal Logics

Proof of Theorem 1

Let $\mathcal{F} = (W, R)$ be a frame. The following statements are equivalent:

1. R is reflexive;

2. $\mathcal{F}$ satisfies $\Box\phi \rightarrow \phi$;

3. $\mathcal{F}$ satisfies $\Box p \rightarrow p$ for any atom p

$1 \Rightarrow 2$: Let R be reflexive. Let L be any labeling function; $\mathcal{M} = (W, R, L)$. Need to show for any x:
$x \Vdash \Box\phi \rightarrow \phi$ Suppose $x \Vdash \Box\phi$.
Since R is reflexive, we have $x \Vdash \phi$.
Using the semantics of $\rightarrow$: $x \Vdash \Box\phi \rightarrow \phi$

Motivation
Basic Modal Logic
Logic Engineering

Valid Formulas wrt Modalities
Properties of R
Correspondence Theory
Preview: Some Modal Logics

Proof of Theorem 1

Let $\mathcal{F} = (W, R)$ be a frame. The following statements are equivalent:

1. R is reflexive;

2. $\mathcal{F}$ satisfies $\Box\phi \to \phi$;

3. $\mathcal{F}$ satisfies $\Box p \to p$ for any atom p

   $2 \Rightarrow 3$: Just set ϕ to be p

Motivation
Basic Modal Logic
Logic Engineering

Valid Formulas wrt Modalities
Properties of R
Correspondence Theory
Preview: Some Modal Logics

Proof of Theorem 1

Let $\mathcal{F} = (W, R)$ be a frame. The following statements are equivalent:

1. R is reflexive;

2. $\mathcal{F}$ satisfies $\Box \phi \rightarrow \phi$;

3. $\mathcal{F}$ satisfies $\Box p \rightarrow p$ for any atom p

    $3 \Rightarrow 1$: Suppose the frame satisfies $\Box p \rightarrow p$.
           Take any world x from W.
           Choose a labeling function L such that $p \notin L(x)$,
           but $p \in L(y)$ for all y with $y \neq x$
           Proof by contradiction: Assume $(x, x) \notin R$. Then
           we would have $x \Vdash \Box p$, but not $x \Vdash p$.
           Contradiction!

Motivation
Basic Modal Logic
Logic Engineering

Valid Formulas wrt Modalities
Properties of R
Correspondence Theory
Preview: Some Modal Logics

Formula Schemes and Properties of R

name	formula scheme	property of R
T	$\Box\phi \rightarrow \phi$	reflexive
B	$\phi \rightarrow \Box\Diamond\phi$	symmetric
D	$\Box\phi \rightarrow \Diamond\phi$	serial
4	$\Box\phi \rightarrow \Box\Box\phi$	transitive
5	$\Diamond\phi \rightarrow \Box\Diamond\phi$	Euclidean
	$\Box\phi \leftrightarrow \Diamond\phi$	functional
	$\Box(\phi \wedge \Box\phi \rightarrow \psi) \vee \Box(\psi \wedge \Box\psi \rightarrow \phi)$	linear

Motivation
Basic Modal Logic
Logic Engineering

Valid Formulas wrt Modalities
Properties of R
Correspondence Theory
Preview: Some Modal Logics

Which Formula Schemes to Choose?

Definition

Let $\mathcal{L}$ be a set of formula schemes and $\Gamma \cup \{\psi\}$ a set of formulas of basic modal logic.

- A set of formula schemes is said to be *closed* iff it contains all substitution instances of its elements.
- Let $\mathcal{L}_c$ be the smallest closed superset of $\mathcal{L}$.
- Γ entails ψ in $\mathcal{L}$ iff $\Gamma \cup \mathcal{L}_c$ semantically entails ψ. We say $\Gamma \models_\mathcal{L} \psi$.

Motivation
Basic Modal Logic
Logic Engineering

Valid Formulas wrt Modalities
Properties of R
Correspondence Theory
Preview: Some Modal Logics

Examples of Modal Logics: K

K is the weakest modal logic, $\mathcal{L} = \emptyset$.

Motivation
Basic Modal Logic
Logic Engineering

Valid Formulas wrt Modalities
Properties of R
Correspondence Theory
Preview: Some Modal Logics

Examples of Modal Logics: KT45

$\mathcal{L} = \{T, 4, 5\}$

Used for reasoning about knowledge.

- T: Truth: agent Q only knows true things.
- 4: Positive introspection: If Q knows something, he knows that he knows it.
- 5: Negative introspection: If Q doesn't know something, he knows that he doesn't know it.

Motivation
Basic Modal Logic
Logic Engineering

Valid Formulas wrt Modalities
Properties of R
Correspondence Theory
Preview: Some Modal Logics

Next Week

- Examples of Modal Logic
- Natural deduction in modal logic