Review of Probability Theory

Sample space and events. Consider the simple experiment of throwing a die. There are six possible outcomes: 1,2,3,4,5, and 6. The *sample space* Ω is the set of all possible outcomes of an experiment:

$$\Omega = \{1, 2, 3, 4, 5, 6\}.$$

An *event* is a subset of the sample space Ω. For example, We might be interested in the folllowing events:

- The outcome is 1. $\Leftrightarrow \{1\}$
- The outcome is strictly greater than 3. $\Leftrightarrow \{4, 5, 6\}$
- The outcome is even. $\Leftrightarrow \{2, 4, 6\}$

Probability Often we are interested in the likelihoods of the occurances of events. A *probability function* maps an event to a real number in $[0, 1]$, which specifies how likely the event occurs. Mathematically a probability satisfies three properties:

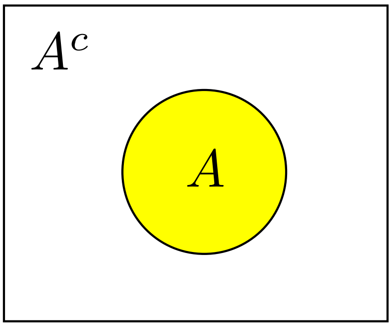
1. $\Pr(\emptyset) = 0$ and $\Pr(\Omega) = 1$.
2. If $A_1, A_2, \ldots$ is a collection of mutually exclusive events, then

$$\Pr\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \Pr(A_i).$$

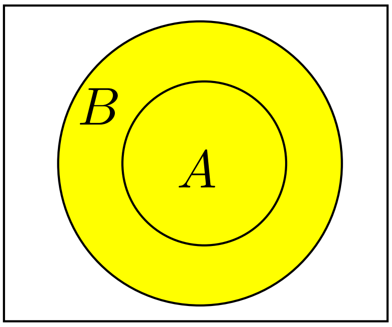
Example Suppose that all outcomes of our experiment are equally likely, *i.e.*, $\Pr(\{i\}) = 1/6$ for all i. Then $\Pr(\{4,5,6\}) = \Pr(\{4\}) + \Pr(\{5\}) + \Pr(\{6\}) = 1/2$.

Important properties of probability

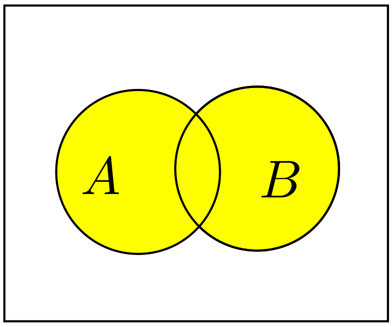
1. $\Pr(A^c) = 1 - \Pr(A)$ for any event A.
2. For two events A and B, if $A \subset B$, then $\Pr(A) \leq \Pr(B)$.
3. For any two events A and B, $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) \leq \Pr(A) + \Pr(B)$.



Property 1



Property 2



Property 3

Conditional probability Many statements about probabilities are concerned with the relationship between two events A and B. They often take the form "if A occurs, what is probability of B occurring"? The *conditional probability* that B occurs given that A occurs is defined to be

$$\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)},$$

if $\Pr(A) > 0$.

Example Consider now the experiment of throwing two dice. Given that the first one shows 3, what is the probability that the total exceeds 6?

- The sample space is $\Omega = \{(a, b)\}$ for $a, b \in \{1, 2, 3, 4, 5, 6\}$.
- Let A be the event that the total sum exceeds 6. We have $A = \{(a, b) \colon a + b > 6\}$.
- Let B be the event that the first die shows 3. We have $B = \{(3, b) \colon 1 \leq b \leq 6\}$.
- $A \cap B = \{(3, 4), (3, 5), (3, 6)\}$.

Thus

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{1/12}{1/6} = \frac{1}{2}.$$

A fundamental lemma *For any events A and B,*

$$\Pr(A) = \Pr(A|B)\Pr(B) + \Pr(A|B^c)\Pr(B^c).$$

case 1 case 2

This formula provides us a way to calculate $\Pr(A)$ using *case anaysis*. More generally, let $B_1, B_2, \ldots, B_n$ be a partition of Ω. Then

$$\Pr(A) = \sum_{i=1}^{n} \Pr(A|B_i)\Pr(B_i).$$

Independence We say that two events A and B are *independent*, if

$$\Pr(A|B) = \Pr(A).$$

Intuitivey it means that A and B have no relationship with each other and that the occurrance of B gives no information about the occurrance of A. From the definition of conditional probability, $\Pr(A \cap B) = \Pr(A|B)\Pr(B)$. If A and B are independent, then

$$\Pr(A \cap B) = \Pr(A)\Pr(B).$$

In fact, the above conditions given in the two equations above are equivalent, provided $\Pr(B) \neq 0$.

For exampe, $\Pr(\{(1,1)\}) = \Pr(\{1\}) \cdot \Pr(\{1\}) = 1/6 \cdot 1/6 = 1/36$. Now if we throw k dice, what is the probability of getting all 1? $1/6^k$. We cannot be unlucky all the time!