Problem Formulation

CS6240 Multimedia Analysis

Leow Wee Kheng

Department of Computer Science
School of Computing
National University of Singapore

Problem Formulation

Before attempting to solve a problem, we need to first formulate or define the problem.

It is important to precisely define the problem you intend to solve.

> *The more difficult it is to define the problem,*
> *the harder you have to try.*

Why?

A Practical Example: Image Mosaicking

(a)

(b)

(c)

How to register images (a) and (b) to produce image (c)?

Maybe, we can write the problem as:

> *Find the transformation between image (a) and image (b), then transform one of them and blend them together.*

This description is not precise.
How to write a program according to the English description?

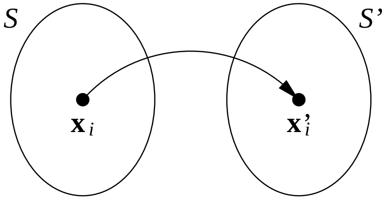
What makes a good problem definition?

- It is precise.
- It states the objectives (what is required):
  - the inputs and their characteristics, including initial conditions
  - the outputs and their desired characteristics, including goal conditions
  - the relationships between the inputs and the desired outputs

How to state a good problem definition?
Answer: Use mathematical notations.

One way to learn how to write good problem definitions is to start with generic problems.

Abstract Mapping Problem



- Let S denote a set $\{\mathbf{x}_i\}$ of n points $\mathbf{x}_i$ called the source.
- Let S' denote a set $\{\mathbf{x}'_i\}$ of n points $\mathbf{x}'_i$ called the target.
- Suppose we know that there is a mapping from each $\mathbf{x}_i \in S$ to $\mathbf{x}'_i \in S'$.
- We want to determine the mapping function.

How to formulate this problem?

A possible problem formulation:

> *Given a set* S *of* n *points* $\mathbf{x}_i$ *and a set* S' *of* n *points* $\mathbf{x}'_i$*, determine the function* $f : S \rightarrow S'$ *such that* $\mathbf{x}'_i = f(\mathbf{x}_i)$ *for* $i = 1, \ldots, n$*.*

“Determine the function” means “determine the form and parameters of the function”.

Example: Linear Case

In the linear case, $\mathbf{x}'_i = f(\mathbf{x}_i)$ can be written in matrix form:

$$\mathbf{x}'_i = \mathbf{F}\,\mathbf{x}_i\,. \tag{1}$$

In this case, the form is a linear equation and the parameters are the values of the matrix elements in $\mathbf{F}$.

In practice, we usually cannot obtain the exact $\mathbf{F}$.
There is an error $e_i = \|\mathbf{x}'_i - \mathbf{F}\mathbf{x}_i\|$.

So, we can re-formulate the problem as:

> *Given a set S of n points $\mathbf{x}_i$ and a set S' of n points $\mathbf{x}'_i$, determine the matrix $\mathbf{F}$ that minimizes the sum-squared error E:*

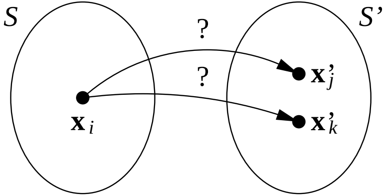
$$E = \sum_{i=1}^{n} \|\mathbf{x}'_i - \mathbf{F}\mathbf{x}_i\|^2. \tag{2}$$

- Now, the problem becomes an optimization problem.
- Eq. 2 is called the objective function.

Generic Mapping Problem

In the previous example, the mapping or correspondence is known.

What if the mapping is unknown?



- Let S denote a set $\{\mathbf{x}_i\}$ of m points $\mathbf{x}_i$.
- Let S' denote a set $\{\mathbf{x}'_i\}$ of n points $\mathbf{x}'_j$, m may or may not be equal to n.

- We know that a point in S can map to some points in S', but don't know which one.
- So, must use different subscripts for points in S and S'. Same subscript implicitly means "known correspondence".

A possible problem formulation:

> *Given a set S of m points $\mathbf{x}_i$ and a set S' of n points $\mathbf{x}'_j$, determine the function $f : S \to S'$ such that for each $\mathbf{x}_i \in S$, there is a $\mathbf{x}'_j \in S'$ such that $\mathbf{x}'_j = f(\mathbf{x}_i)$.*

- This formulation is ambiguous.
- There are many possible f. Which one is it talking about?

Suppose we know how to measure the difference d between any $\mathbf{x}_i \in S$ and $\mathbf{x}'_j \in S'$.

We can formulate the problem as one of finding the best mapping:

> *Given a set S of m points $\mathbf{x}_i$, a set S' of n points $\mathbf{x}'_j$, and a difference measure $d(\mathbf{x}_i, \mathbf{x}'_j)$, determine the function $f : S \rightarrow S'$ that minimizes the sum-squared error E:*

$$E = \sum_{i=1}^{m} d^2(\mathbf{x}_i, f(\mathbf{x}_i)) \,. \tag{3}$$

Questions:

- In the above formulation, why use $d(\mathbf{x}_i, f(\mathbf{x}_i))$ instead of $d(\mathbf{x}_i, \mathbf{x}'_j)$?
- Can we use $d(\mathbf{x}'_j, f(\mathbf{x}_i))$ as in Eq. 2?
- What are the differences between this problem definition and the one in Eq. 2?

A Practical Example: Image Mosaicking

This problem can be divided into three sub-problems:

1. Identify corresponding points between the two images.
2. Compute transformation between the two images.
3. Transform and blend images.

Let's consider Sub-Problem 2.

Suppose the transformation $\mathbf{T}$ is linear.

Then, ideally

$$\mathbf{x}'_i = \mathbf{T}\mathbf{x}_i, \text{ for each point } i. \tag{4}$$

In reality, there is an error $e_i = \|\mathbf{x}'_i - \mathbf{T}\mathbf{x}_i\|$.

So, we can formulate the problem as follows:

> *Given a set S of n points $\mathbf{x}_i$ and a set S' of n points $\mathbf{x}'_i$, determine the matrix $\mathbf{T}$ that minimizes the sum-squared error E:*

$$E = \sum_{i=1}^{n} \|\mathbf{x}'_i - \mathbf{T}\mathbf{x}_i\|^2. \tag{5}$$

Note:

- To completely define image mosaicking problem, need to define sub-problems 1 and 3 as well. (Exercise)

Constrained Mapping Problem

In some applications, there are constraints that must be satisfied.

Consider the generic mapping problem in Section 1:

- The problem definition does not prohibit multiple points in S to map to a single point in S', i.e., it allows for many-to-one mapping.
- Suppose we need to impose one-to-one mapping.
- Then, the problem definition can be re-formulated as follows:

Given a set S of m points $\mathbf{x}_i$, a set S' of n points $\mathbf{x}'_j$, and a difference measure $d(\mathbf{x}_i, \mathbf{x}'_j)$, determine the function $f : S \to S'$ that minimizes the sum-squared error E:

$$E = \sum_{i=1}^{m} d^2(\mathbf{x}_i, f(\mathbf{x}_i)) \tag{6}$$

subject to the constraint that f is a one-to-one function.

Another way to describe the constraint is:

> *subject to the constraint that* $f(\mathbf{x}_i) \neq f(\mathbf{x}_k)$ *for any* $\mathbf{x}_i, \mathbf{x}_k \in S$ *such that* $\mathbf{x}_i \neq \mathbf{x}_k$.

The above problem is called a constrained optimization problem.

Summary

- A good problem definition is precise and it states the problem requirements and objectives.
- Many multimedia analysis problems can be formulated as optimization problems.
- Before you try to solve a problem, first study it carefully and then formulate the problem.
- Usually, you need to revise your problem formulations several times to make it more precise and more correctly describe the problem.

Exercise

(1) Define sub-problems 1 and 3 of image mosaicking.