NATIONAL UNIVERSITY OF SINGAPORE

SCHOOL OF COMPUTING
SEMESTER II: 2005–2006
EXAMINATION FOR
GEM 1501 – Problem Solving for Computing
Friday 21 April 2006 Morning – Time Allowed 2 Hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of FIFTEEN (15) questions and comprises NINE (9) printed pages.
2. Answer **ALL** questions.
3. This is an **Closed Book** examination.
4. Every question counts TWELVE (12) Marks which is distributed equally on subquestions in the case that there are any. The maximum possible marks are 180.
5. Please write your Matriculation Number below:

MATRICULATION NO: ______________________________

This portion is for examiner's use only

Question	Marks	Remarks	Question	Marks	Remarks	Question	Marks	Remarks
Q01:			Q06:			Q11:		
Q02:			Q07:			Q12:		
Q03:			Q08:			Q13:		
Q04:			Q09:			Q14:		
Q05:			Q10:			Q15:		
						Total:		

Question 1 [12 marks]. The following program for a function f needs time exponential in n to compute $f(n)$. The input n is a natural number.

```
function f(n)
  { var m = n;
    if (n > 3) { m = f(n-1)+f(n-3)+f(Math.floor(0.5*n)); }
    return(m); }
```

Write a better program doing it in $O(n)$.

```
function f(n)
  { var a = new Array(0,1,2);
    while (a.length <= n)
      { a.push(a[a.length-1]+a[a.length-3]+a[Math.floor(0.5*a.length)]); }
    return(a[n]); }
```

Question 2 [12 marks]. Consider the following program.

```
function g(n)
  { var m = 0; var a=0; var b=1; var c;
    while (a<n) { c=a+b; a=b; b=c; m=m+1; }
    return(m); }
```

What is the value $g(8)$? 6

What is the value $g(13)$? 7

Is the value 2 in the range of g? □ Yes, ☒ No.

The function is the inverse of a well-known mathematical sequence. What is the name for the numbers of this sequence in mathematics? Fibonacci Numbers

What happens if the function is between two members of the sequence?

□ It does not terminate; □ It takes the lower value; ☒ It takes the higher value.

What is the order of the algorithm measured in the actual input n (not measured in the number of digits of n):

□ $O(\log\log(n))$; ☒ $O(\log(n))$; □ $O(n)$; □ $O(2^n)$; □ $O(2^{2^n})$.

Question 3 [12 marks]. Correct in the following program all syntax errors. Each line has one of them.

```
INCORRECT PROGRAM                            CORRECT PROGRAM
functional h(x,y,z)                          function h(x,y,z)
  { this.name = x                              { this.name = x;
    this.code = y                                this.code = y;
    this.bl = z }                                this.bl = z ; }
functional hwrite(c)                         function hwrite(c)
  { documentwrite("Name: "+c.name+"; ");       { document.write("Name: "+c.name+"; ");
    documentwrite("Code: "+c.code+"; ");         document.write("Code: "+c.code+"; ");
    documentwrite("Blk: "+c.bl+".<br>"); }       document.write("Blk: "+c.bl+".<br>"); }
variable a = new h("SoC", 117543,"S16");      var a = new h("SoC", 117543,"S16");
variable b = new h("Math",117543,"S14");      var b = new h("Math",117543,"S14");
if (a.code == b.code) [                       if (a.code == b.code) {
  hwrite(a); hwrite(b); ]                       hwrite(a); hwrite(b); }
```

Question 4 [12 marks]. What is an array? An array is a one-dimensional or multidimensional list of similar data items.

What is a record? A record is an object consisting of a fixed number of entries which have names. In Javascript it can be viewed as an array of fixed size where all array members also have a name.

How can two-dimensional arrays be implemented in Java Script? One can make an one-dimensional array and then define for each field in the array, that it is a one-dimensional array itself.

Write a function which produces a record with fields called "numbername" and "samplenumber" and then produce an array using this function which contains the three entries ("prime",17), ("square",25), ("perfect",28).

```
function record(x,y)
  { this.numbername = x;
    this.samplenumber = y; }
var numbers = new Array(3);
numbers[0] = new record("prime",17);
numbers[1] = new record("square",25);
numbers[2] = new record("perfect",28);
```

Question 5 [12 marks]. Find for each of the following statements the corresponding language to which it applies most.

This language was developed more for education purposes than industrial programming and still exists in several variants like as a built-in programming language for macros of the text system Word.

□ APL, ☒ Basic, □ C, □ Fortran □ Java, □ SNOBOL.

This was one of the first programming languages, but its current form does not resemble much to its beginnings.

□ APL, □ Basic, □ C, ☒ Fortran □ Java, □ SNOBOL.

This language permits to write fast executable code and has specific operators like "+=", "++" and "−−" permitting to do several operations at once when accessing a variable. Features like these were copied into many more recent programming languages.

□ APL, □ Basic, ☒ C, □ Fortran □ Java, □ SNOBOL.

This language was mainly used in string processing but is no longer very popular in that field.

□ APL, □ Basic, □ C, □ Fortran □ Java, ☒ SNOBOL.

This language has a lot of matrix and tensor operators which have specific symbols. Therefore its user interface included a specific typesetting program to type these symbols on the keyboard.

☒ APL, □ Basic, □ C, □ Fortran □ Java, □ SNOBOL.

This language is frequently used for programming webpages and became quite popular in the recent years.

□ APL, □ Basic, □ C, □ Fortran ☒ Java, □ SNOBOL.

Question 6 [12 marks]. Given is a square matrix a such that each field $a[i][j]$ contains a 1 if one can go from i to j directly and a 0 otherwise. Furthermore, there might be indirect ways to go from i to j. If there is an ℓ such that $a[i][\ell]$ and $a[\ell][j]$ are both 1 then one can go from i through this ℓ to j. Similarly one could also go indirectly through several nodes. Complete the below algorithm such that it does the following: It initializes an array b as $(1, 0, 0, \ldots, 0)$ consisting of as many elements as a has rows (or columns) and then runs the loops to make $b[j]$ to 1 for all those j which can be reached from 0 by direct or indirect travel.

```
function reach(a)
  { var i; var j; var k; var h=a. length ;
    var b = new Array();
    for (k=0;k<h;k=k+1)
      { b. push (0); }
    if (h>0) { b[0]=1; }
    for (k=0;k<h;k=k+1)
      { for (i=0;i<h;i=i+1)
          { for (j=0;j<h;j=j+1)
              { if ( (b[i]*a[i][j]==1 )
                   { b[j]= 1 ; } } } }
    return(b); }
```

How is this type of algorithm called?

□ Divide and Conquer; ☒ Greedy; □ Dynamic Programming.

Determine the order of this algorithm in dependence of the number n of rows of the matrix a:

□ $O(n)$; □ $O(n^2)$; ☒ $O(n^3)$; □ $O(n^4)$; □ $O(2^n)$; □ $O(3^n)$; □ $O(4^n)$.

Question 7 [12 marks]. Answer the following questions on terminology and basic facts.

If a partially correct program always terminates, is it then correct? ☒ Yes, □ No.
Are there programs which are partially correct but not correct? ☒ Yes, □ No.
What word is used for a formal description of what a program has to do?
□ Compilation; □ Interpretation; ☒ Specification; □ Verification.
A Turing machine is □ another word for a finite state machine;
☒ a simple mathematical model of a computer;
□ a machine used by the British in World War II to decrypt German messages;
□ an improved version of Babbage's Analytical Engine;
□ a term from some science-fiction book without any relation to computer science.
What is the order of the average case complexity of Bubble Sort:
□ $O(n \log\log(n))$; □ $O(n \log(n))$; □ $O(n \log^2(n))$; ☒ $O(n^2)$; □ $O(n^2 \log(n))$.
What is the order of the worst case complexity of Merge Sort:
□ $O(n \log\log(n))$; ☒ $O(n \log(n))$; □ $O(n \log^2(n))$; □ $O(n^2)$; □ $O(n^2 \log(n))$.

Question 8 [12 marks]. Which of the following rules on the order of functions are true? The functions f, g are from the set $\{0, 1, 2, 3, \ldots\}$ of natural numbers to itself.

If $\forall n\,(f(n) < 3g(n))$ then $f \in O(g)$:	☒ Yes,	□ No.
If $f \in O(g)$ then $O(f+g) = O(g)$:	☒ Yes,	□ No.
If $f \in O(g)$ then $g \in O(f)$:	□ Yes,	☒ No.
If $f \in O(g)$ then $2^f \in O(2^g)$:	□ Yes,	☒ No.
$1 + 4n + 6n^2 + 4n^3 + n^4 \in O(n^5)$:	☒ Yes,	□ No.
$n \log\log(n) + n^3 \in O(n^2 \log(n))$:	□ Yes,	☒ No.
$O(3n^3 \log^2(n) + n^4) = O(n^4 \log^2(n))$:	□ Yes,	☒ No.
If $\forall n\,(f(n) = f(n+1))$ then $O(f) = O(5)$:	□ Yes,	☒ No.
If $f \in O(\log(n))$ then $f \in O(n)$:	☒ Yes,	□ No.
If $f \in O(g)$ and $g \in O(f)$ then $O(f) = O(g)$:	☒ Yes,	□ No.
If $\forall n\,(f(n) \leq n)$ then $O(f) = O(n)$:	□ Yes,	☒ No.
If $O(f) \subseteq O(g)$ then $f \in O(g)$:	☒ Yes,	□ No.

Question 9 [12 marks]. Consider the following function, the inputs x, y are natural numbers.

```
function calc(x,y)
  { if (y<1) { return(1); }
    if (x<y) { return(0); }
    var i; var j=0;
    for (i=0;i<x;i=i+1)
      { j = j+2*calc(i,y-1); }
    return(j); }
```

Analyze this function and give easy formulas for the following derived functions `fa,fb,fc,fd`:

- `fa(n) = calc(n,n)` is 2^n;
- `fb(n) = calc(n,1)` is $2 * n$;
- `fc(n) = calc(n,2)` is $2 * n * n - 2 * n$;
- `fd(n) = calc(n,0)+calc(n,1)+...+calc(n,n)` is 3^n.

Question 10 [12 marks]. Consider the following function for the Towers of Hanoi with four pegs where the command `move(ta,tb)` moves one ring from the peg `ta` to the peg `tb`, `n` is the number of rings and `m` is a suitably chosen parameter. The names of the pegs are 0,1,2,3 and the tower is moved from Peg 0 to Peg 1.

```
function movefour(n,m,ta,tb,tc,td)
  {
  if (n > m)
    { movefour(n-m,m,ta,tc,tb,td);
      movefour(m,m,ta,tb,tc,td);
      movefour(n-m,m,tc,tb,ta,td); }
  else if (n > 1)
    { movefour(n-1,m,ta,td,tc,tb);
      move(ta,tb);
      movefour(n-1,m,td,tb,tc,ta); }
  else if (n == 1) { move(ta,tb); }
  return;
  }
movefour(n,m,0,1,2,3);
```

In the case that `n` is the product of `m` and `k` which are both positive integers, what is the number of moves made by this function? $(2^m - 1) * (2^k - 1)$.
Write a piece of Java Script function which chooses `m` optimally or nearly optimally in dependence of `n` for this algorithm; `m` should not be more off than `5` from the optimum for any `n`. You can define additional variables and loops to compute `m` if needed.

```
m = Math.round(Math.sqrt(n));
```

What is the order of the number of moves of the overall algorithm if `m` is chosen optimally?

□ $O(n)$; □ $O(n^2)$; □ $O(n^3)$; □ $O(n^4)$; □ $O(n^5)$;
□ $O(1^{\sqrt{n}})$; □ $O(2^{\sqrt{n}})$; □ $O(3^{\sqrt{n}})$; ☒ $O(4^{\sqrt{n}})$; □ $O(5^{\sqrt{n}})$;
□ $O(2^{n/1})$; □ $O(2^{n/2})$; □ $O(2^{n/3})$; □ $O(2^{n/4})$; □ $O(2^{n/5})$;
□ $O(2^n/n)$; □ $O(2^n/n^2)$; □ $O(2^n/n^3)$; □ $O(2^n/n^4)$; □ $O(2^n/n^5)$.

Question 11 [12 marks]. Concerning complexity classes, many things are still unknown, for example no one knows whether P ⊂ NP is true or false. One only knows that P ⊆ NP. Please identify for the following statements whether they are currently unknown, true or false.

NC ⊂ EXPTIME: □ Unknown, ☒ Yes, □ No.
LOGSPACE ⊂ POLYLOGSPACE: □ Unknown, ☒ Yes, □ No.
PSPACE ⊂ P: □ Unknown, □ Yes, ☒ No.
NP ⊂ EXPTIME: ☒ Unknown, □ Yes, □ No.
NC ⊂ LOGSPACE: □ Unknown, □ Yes, ☒ No.
NC ⊂ POLYLOGSPACE: ☒ Unknown, □ Yes, □ No.

Question 12 [12 marks]. Recall that every instance of a problem can be written down formally as a text following some fixed converentions. Furthermore, the class P is the class of class of all sets A of texts such that there is a deterministic Turing-machine M and a polynomial p such that for every text T of length n, the machine M halts on input T within $p(n)$ steps with output 1 if $T \in A$ and with output 0 if $T \notin A$. Write the corresponding definitions of the following complexity classes in a similar formal way; you can consider other machine models than Turing machines if adequate.

LOGSPACE: The class LOGSPACE is the class of class of all sets A of texts such that there is a deterministic Turing-machine M and a linear function p such that for every text T of length n, the machine M halts on input T using space $p(\log(n))$ with output 1 if $T \in A$ and with output 0 if $T \notin A$.

POLYLOGSPACE: The class POLYLOGSPACE is the class of class of all sets A of texts such that there is a deterministic Turing-machine M and a polynomial p such that for every text T of length n, the machine M halts on input T using space $p(\log(n))$ with output 1 if $T \in A$ and with output 0 if $T \notin A$.

NC: The class NC is the class of all sets A of texts such that there is a deterministic Turing-machine M and a polynomial p such that M computes in time $p(n)$ on input n a circuit C consisting of up to $p(n)$ gates and depth up to $p(\log(n))$ such that $T \in A$ if and only if the circuit C outputs a 1 on a binary representation of T as input.

RP: The class RP is the class of class of all sets A of texts such that there is a nondeterministic Turing-machine M and a polynomial p such that for every text T of length n, the machine M halts on input T within $p(n)$ steps such that the output is 1 for at least one half of the possible computations if $T \in A$ and with output 0 for all possible computations if $T \notin A$.

Question 13 [12 marks]. For the following sets, give the complexity class which would apply under the assumption that P, NP and PSPACE are different (without this assumption, the solution might not be unique). Use "intractable" if there is an algorithm but no known one in PSPACE. Make only one check per line.

Monkey Puzzle for fixed area:
□ P, ☒ NP-complete, □ PSPACE-complete, □ Intractable, □ Undecidable.

Monkey Puzzle for all areas:
□ P, □ NP-complete, □ PSPACE-complete, □ Intractable, ☒ Undecidable.

Set of pairs of numbers where the greatest common divisor is not 1:
☒ P, □ NP-complete, □ PSPACE-complete, □ Intractable, □ Undecidable.

Evaluating formulas of the form $\exists x_1 \forall y_1 \exists x_2 \forall y_2 \ldots \exists x_n \forall y_n \Phi(x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$ where the variables take Boolean values, Φ is a Boolean formula and n is not constant:
□ P, □ NP-complete, ☒ PSPACE-complete, □ Intractable, □ Undecidable.

"2SAT"-variant of the Satisfiability Problem:
☒ P, □ NP-complete, □ PSPACE-complete, □ Intractable, □ Undecidable.

"4SAT"-variant of the Satisfiability Problem:
□ P, ☒ NP-complete, □ PSPACE-complete, □ Intractable, □ Undecidable.

Question 14 [12 marks]. Write a finite automaton which tests whether a binary number which might have leading zeroes is a multiple of ten. If the number is not a multiple of ten, the automaton should accept, otherwise it should reject. So it should accept the inputs 10, 001, 1001, 1101 but reject the empty input, 0, 1010, 10100 and 01010. Note explicitly the starting state and the set of accepting states. Fill out the following table. Of the below lines, use as many as needed. Only "0" and "1" occur as input-symbols.

Statename	Successor at 0	Successor at 1	Accepting/Rejecting
s0	s0	s1	Rejecting
s1	s2	s3	Accepting
s2	s4	s5	Accepting
s3	s6	s7	Accepting
s4	s8	s9	Accepting
s5	s0	s1	Accepting
s6	s2	s3	Accepting
s7	s4	s5	Accepting
s8	s6	s7	Accepting
s9	s8	s9	Accepting

The starting state is s0 .

Question 15 [12 marks]. Presburger proved that one can decide whether a finite system of conditions on finitely many variables consisting of integer constants, additions, substractions and comparisons has a solution in the integers. Matiyasevich proved that is impossible if multiplication is also permitted. But what happens if one considers a finite system of conditions on finitely many variables consisting of integer constants, additions, substractions, squaring and comparisons? An example of such a system would be the following:

- $x_1 \geq (x_2 + x_3 - 8)^2$;
- $x_6 > (x_1 + x_2)^2 - (x_1 - x_2)^2$;
- $x_4 \leq (x_2 + x_3 - 9)^2$;
- $x^5 \geq ((x_1 + 8)^2 + 8)^2$.

This system has solutions, for example x_1 is 5, x_2 is 5, x_3 is 5, x_4 is -1, x_5 is 40000 and x_6 is 100. Is it decidable (by an algorithm) whether such a system has a solution in the integers? Give an answer and prove it.

This is undecideable as one can code diophantine sets. Given a diophantine set

$$D = \{x : \exists y_1, y_2, \ldots, y_9\, [p(x, y_1, y_2, \ldots, y_9) = 0]\},$$

one can first multiply the polynomial p with a constant 2^c where c is the number of multiplications in it and move factors into the polynomial such that in the resulting new polynomial formula every multiplication is either of the form $2 * v * w$ or of the form $2 * u$. Those of the form $2 * u$ are replaced by $u + u$. Multiplications of the form $2 * v * w$ are replaced by $(v + w)^2 - v^2 - w^2$ which is equivalent to $v^2 + w^2 + 2 * v * w - v^2 - w^2$. Let q be the new formula which only uses additions, substractions and squarings. Then

$$D = \{x : \exists y_1, y_2, \ldots, y_9\, [q(x, y_1, y_2, \ldots, y_9) = 0]\},$$

and so the question whether such a q admits a solutino is undecidable.

END OF PAPER