Midterm Examination 2
GEM 1501: Problem Solving for Computing

09.04.2008, 12.00-12.30h

Matriculation Number: _____

Rules
Each correct question, 1 mark. Maximum score: 12 marks.
Programming Language for Questions 7–11 is Java Script.

Question 1. Solve the following problem using resolution. Do each step as prescribed. The logical variables are x_1, x_2, x_3, x_4, x_5, write the clauses after each step.

Given clauses: $\neg x_1 \vee x_2$, $x_1 \vee \neg x_3$, $x_1 \vee \neg x_4$, $x_3 \vee x_5$, $\neg x_4 \vee \neg x_5$, $x_4 \vee x_5$.

Resolve x_1, new clauses:
$x_2 \vee \neg x_3$, $x_2 \vee \neg x_4$, $x_3 \vee x_5$, $\neg x_4 \vee \neg x_5$, $x_4 \vee x_5$.

Choose x_2 as ☐ false ☒ true, new clauses:
$x_3 \vee x_5$, $\neg x_4 \vee \neg x_5$, $x_4 \vee x_5$.

Choose x_3 as ☐ false ☒ true, new clauses:
$\neg x_4 \vee \neg x_5$, $x_4 \vee x_5$.

Resolve x_4, new clause: $\neg x_5 \vee x_5$.

This instance of the 2SAT problem is ☒ satisfiable ☐ not satisfiable.

Question 2. Presburger Arithmetic permits formulas over variables for integers using the addition, the relations $<$, $\leq$, $=$ and integer constants. Which of the following formulas is true in Presburger Arithmetic?

☐ $\forall x \, \exists y \, [x + y > y + x]$;
☒ $\forall x \, \exists y \, [x + y = 0]$;
☐ $\forall x \, \exists y \, [y + y = x]$.

Furthermore, what is the complexity of Presburger Arithmetic:

☐ NC (Nick's Class) ☐ P ☐ NP-complete ☐ Exponential time
☒ Double Exponential time ☐ Recursively enumerable and undecidable.

Question 3. Which one of the following three languages is recognized by a one-stack automaton but not by a finite automaton:

☐ $\{w : w$ has as many 0s as 1s as 2s$\}$;

☒ $\{w : w$ has more 0s than 1s$\}$;

☐ $\{w : w$ has at least five 0s and an even number of 1s$\}$.

Question 4. Complete the following definition of Nick's Class.
A problem A is in NC iff there is a polynomial time computable function Cc which produces for every $n > 1$ a circuit $Cc(n)$ such that $Cc(n)$ has n input bits and there is a constant c such that the circuit $Cc(n)$ has at most

☐ $c\log(n)$ ☐ $\log^c(n)$ ☐ $n \cdot c$ ☒ n^c ☐ 2^{n^c}

gates which are arranged in

☐ $c\log(n)$ ☒ $\log^c(n)$ ☐ $n \cdot c$ ☐ n^c ☐ 2^{n^c}

layers (levels) such that $Cc(n)$ outputs 1 if $x \in A$ and 0 if $x \notin A$ for all inputs x of length n.

Note: Lengths $n = 0, 1$ are not considered in above formulas in order to keep notation simple.

Question 5. In parallel computing, semaphors are objects which are used to coordinate the communication between processes which run in parallel. What property is essential for a semaphore:

☐ It can take three values called "green", "yellow" and "red".

☒ Every process can read and (depending on the previous value) update the value of the semaphore in an atomic operation not interupted by other processes.

☐ It is an array with one entry per process where each process can modify the entry belonging to it without the others to interfere.

☐ Semaphores take only two values "0" and "1" and these values signify "process is running" and "process has terminated", respectively.

Question 6. Assume that on a chip one can add and multiply numbers in one single gate and step. Furthermore, the inputs of the chip are numbers $a_1, a_2, \ldots, a_n$ and it should compute the function

$$f(a_1, a_2, \ldots, a_n) = \sum_{i,j \in \{1,2,\ldots,n\}} a_i^2 a_j^2.$$

How many layers are needed, that is, what is the parallel time complexity?

☐ $O(1)$, ☒ $O(\log(n))$, ☐ $O(\log^2(n))$, ☐ $O(n)$, ☐ $O(n\log(n))$.

Mark the optimal answer, select exactly one of these choices.

Reason: $\log(n^2)$ levels are the same as $2\log(n)$ levels, hence $O(\log(n))$.

Question 7. Insert the JavaScript commands "window.alert", "window.confirm" and "window.prompt" (each at most twice) into the following program where adequate.

```
var a,b,x,y;
do {
  x = window.prompt("Input x");

  y = window.prompt("Input y");

  a = 0.5*x+0.5*y;

  window.alert("The arithmetic mean of "+x+" and "+y+" is "+a+".");

  b = Math.sqrt(x*y);

  window.alert("The geometric mean of "+x+" and "+y+" is "+b+"."); }

  while (window.confirm("Do you want to go on? "));
```

Question 8. Complete the following program which computes x^y modulo p in polynomial time. Note that due to the use of the modulo (= remainder) the value is restricted to $\{0, 1, 2, \ldots, p-1\}$, so 2^{10} modulo 5 is just 4. The inputs x, y, p are positive natural numbers with $p > 1$.

```
function power(x,y,p)
  { var u = 1; var v = x; var w = y;
    while (w > 0)
      { if (w%2==1) { u = u*v % p; }
        v = v*v % p;
        w = Math.floor(0.5*w); }
    return(u); }
```

Question 9. This function has syntax errors in lines 2 (k=0 to 230) and 3 (comma) .

```
function f(x,y) { var h,k; var i=0; var j=0; // line 1
    for (k=0 to 230; k++)                    // line 2
      { i+=k*x, j+=k*y; }                    // line 3
    return((i&j)%17); }                      // line 4
```

Question 10. Consider the following function "randomsort":

```
function randomsort(a)
  { var n = a.length; var i,j,k; var m = 100*n*n*n*n;
    if (n<2) return;
    while (m>0)
      { m--;
        i = Math.floor(Math.random()*n);
        j = Math.floor(Math.random()*n);
        if ((i<j)&&(a[i]>a[j]))
          { m = 100*n*n*n*n;
            k=a[i]; a[i]=a[j]; a[j]=k; } }
    return; }
```

Which two of the following statements are true for this function?

- ☐ At worst case random choices and inputs, the complexity is $O(n^2)$.
- ☐ At worst case random choices and inputs, the complexity is $O(n^4)$.
- ☒ At worst case random choices and inputs, the complexity is $O(n^6)$.
- ☐ At worst case random choices and inputs, the complexity is $O(n^8)$.
- ☐ At worst case random choices and inputs, the function does not terminate.
- ☐ Whenever the function terminates, the array is sorted.
- ☒ The function returns a non-sorted array with small probability.

Question 11. What function is computed by the following program?

```
function f(n)
  { var m = 0; var k = n;
    while (k>0) { k=k-1; m = m+k+k+1; }
    return(m); }
```

The input n is always a natural number $(0, 1, 2, 3, \ldots)$. Tick the correct formula.

☐ $f(n) = 1$; ☐ $f(n) = 2$; ☐ $f(n) = 3$; ☐ $f(n) = 4$;
☐ $f(n) = n$; ☐ $f(n) = 2n$; ☐ $f(n) = 3n$; ☐ $f(n) = 4n$;
☒ $f(n) = n^2$; ☐ $f(n) = n^3$; ☐ $f(n) = n^4$; ☐ $f(n) = n^5$;
☐ $f(n) = 2^n$; ☐ $f(n) = 3^n$; ☐ $f(n) = 4^n$; ☐ $f(n) = 5^n$;
☐ $f(n) = n^n$; ☐ $f(n) = n^{n^n}$; ☐ $f(n) = n^{n^{n^n}}$; ☐ $f(n) = n^{n^{n^{n^n}}}$.

Hint: simulate the function for some small inputs n in order to see which of the options is the correct one.

Computations show that the first values of f are $0, 1, 4, 9, 16, 25$.

Question 12. Write a function which counts how many members of the array a are equal to the word w. Note that a word is a string and can be compared with other stings using the JavaScript comparator "==" in the same way as it is done with numbers.

```
function wordcount(a,w)
  { var n = 0; var m;
    for (m=0;m<a.length;m=m+1)
      { if (a[m] == w)
          { n = n+1; } }
    return(n); }
```

A student found the following solution which employs that "$a[i]$" is false if $i \geq a.length$ and hence the search aborts then. This solution passed the test below and was therefore counted.

```
Program of student:
function wordcount(a,w)
  { var n=0; var i;
    for (i=0;a[i];i++)
      { if (a[i]==w) { n++; } }
    return(n); }

Testroutine:
var a = new Array("This","is","a","list","which","is","9","words","long");
var b = wordcount(a,"are");
document.write("Wordcount for word 'are' in ("+a+") is "+b+".<br>");
b = wordcount(a,"is");
document.write("Wordcount for word 'is' in ("+a+") is "+b+".<br>");
b = wordcount(a,"which");
document.write("Wordcount for word 'which' in ("+a+") is "+b+".<br>");


Output:
Wordcount for word 'are' in (This,is,a,list,which,is,9,words,long) is 0.
Wordcount for word 'is' in (This,is,a,list,which,is,9,words,long) is 2.
Wordcount for word 'which' in (This,is,a,list,which,is,9,words,long) is 1.
```