On Trees without Hyperimmune Branches

Keng Meng Ng, NTU, Singapore

Frank Stephan, NUS, Singapore

Yue Yang, NUS, Singapore

Liang Yu, NJU, Nanjing

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Overview I

Theorem. There is a co-r.e. tree $\mathbf{T}$ without finite branches such that it has $\mathbf{2^{\aleph_0}}$ many infinite branches $\mathbf{A}$ such that all $\mathbf{A}$ are hyperimmune-free, generalised low, Schnorr trivial, of minimal Turing degree and jump traceable.

Co-r.e. tree: Downward closed subset of binary strings such that the set of strings outside it is recursively enumerable. Branches are finite or infinite strings such that all finite prefixes of them are in the tree.

$\mathbf{A}$ is Hyperimmune-free: Every $\mathbf{A}$-recursive function is majorised by a recursive function.

$\mathbf{A}$ is Schnorr-trivial: For every $\mathbf{f} \leq_{\mathbf{tt}} \mathbf{A}$ there is a recursive function $\mathbf{g}$ such that $\forall \mathbf{n}\, \exists \mathbf{m} \leq \mathbf{n}\, [\mathbf{f}(\mathbf{n}) = \mathbf{g}(\mathbf{n}, \mathbf{m})]$.

$\mathbf{A}$ is Jump Traceable: There are recursive function $\mathbf{f}, \mathbf{g}$ such that for all $\mathbf{e}$, $\varphi_{\mathbf{e}}^{\mathbf{A}}(\mathbf{e}) \in \mathbf{W}_{\mathbf{f}(\mathbf{e})}$ and $|\mathbf{W}_{\mathbf{f}(\mathbf{e})}| \leq \mathbf{g}(\mathbf{e})$.

Overview II

$\mathbf{A}$ is Generalised Low: $\mathbf{A}' \leq \mathbf{A} \oplus \mathbf{K}$. This is a consequence of $\mathbf{A}$ being jump traceable: One considers the partial function $\varphi^{\mathbf{A}}_{\mathbf{s(e)}}(\mathbf{x})$ which outputs for all $\mathbf{x}$ the time until $\varphi^{\mathbf{A}}_{\mathbf{e}}(\mathbf{e})$ converges and then one determines using $\mathbf{K}$ the largest value $\mathbf{t}$ in $\mathbf{W}_{\mathbf{f(s(e))}}$ (if any). Now one can simulate the enumeration of $\mathbf{e}$ into $\mathbf{A}'$ for up to $\mathbf{t}$ steps and decide the jump.

Set of infinite branches is uncountable: This is shown by showing that the double jump of these infinite branches coincides with the upper cone above $\mathbf{K}'$.

Every node of the tree is on an isolated infinite branch: If this would not be true then one could from such a node $\sigma_{\mathbf{0}}$ onwards find using $\mathbf{K}$ nodes $\sigma_{\mathbf{1}}, \sigma_{\mathbf{2}}, \ldots$ such that $\sigma_{\mathbf{k+1}}$ is not a prefix of $\varphi_{\mathbf{k}}$ and $\sigma_{\mathbf{k}} \prec \sigma_{\mathbf{k+1}}$, these nodes then define a $\mathbf{K}$-recursive but not recursive infinite branch which is not hyperimmune-free.

Proposition on Trees

Proposition. If $\mathbf{T}$ is a $\mathbf{K}'$-recursive tree, then $\mathbf{T}$ is contained in a $\mathbf{K}$-r.e. tree $\mathbf{T}'$ such that the infinite branches of $\mathbf{T}'$ and $\mathbf{T}$ coincide.

Proof. This is a folklore result which holds relative to any oracle, but it is needed here relative to $\mathbf{K}$. By the limit lemma, there is a $\mathbf{K}$-recursive approximation $\mathbf{T_s}$ such that for all $\mu \in \mathbf{T}$, $\mathbf{T_s}(\mu) = \mathbf{T}(\mu)$ for almost all $\mathbf{s}$. Now one enumerates a node μ into $\mathbf{T}'$ iff there is a stage $\mathbf{s} > |\mu|$ such that all prefixes ν of μ are in $\mathbf{T_s}$. If $\mathbf{A}$ is an infinite branch of $\mathbf{T}$ then clearly all prefixes of $\mathbf{A}$ are enumerated into $\mathbf{T}'$ else there is a prefix μ of $\mathbf{A}$ which is not in $\mathbf{T}$ and so there is a stage $\mathbf{t}$ such that $\mathbf{T_s}(\mu) = \mathbf{0}$ for all $\mathbf{s} \geq \mathbf{t}$ and then no prefix of $\mathbf{A}$ longer than $|\mu| + \mathbf{t}$ will ever be enumerated into $\mathbf{T}'$.

Genericity

Proposition [Based on Friedberg Jump Inversion Theorem and also folklore]. There is a perfect $\mathbf{K'}$-recursive tree $\mathbf{T}$ such that each infinite branch of it is $\mathbf{2}$-generic.

Proof. Given a set $\mathbf{B} \subseteq \mathbb{N}$ and oracle $\mathbf{K'}$, one starts with the root $\sigma_\mathbf{0} = \varepsilon$ and puts it into the tree. Then one checks whether there is an extension of $\sigma_\mathbf{n}\mathbf{B}(\mathbf{n})$ in $\mathbf{W_e^K}$ and if so then one lets $\sigma_\mathbf{n+1}$ be the first such extension found by search relative to $\mathbf{K}$ else $\sigma_\mathbf{n+1}$ is just $\sigma_\mathbf{n}\mathbf{B}(\mathbf{n})$; this case-distinction is clearly $\mathbf{K'}$-recursive. The resulting tree $\mathbf{T}$ is the union of this construction for all oracles $\mathbf{B}$ and $\mathbf{T}[\mathbf{B}]$ refers to the encoded infinite branch for $\mathbf{B}$.

Schnorr Trivial

Remark. Schnorr trivial plus hyperimmune-free implies recursively traceable; however, the property Schnorr trivial is easier to achieve in a construction. Furthermore, if $\varphi_{\mathbf{e}}^{\mathbf{A}}$ is a truth-table reduction then it is assumed that there is a recursive function $\mathbf{g}$ such that $\varphi_{\mathbf{e}}^{\mathbf{A}}(\mathbf{x})$ needs, independently of the choice of $\mathbf{A}$, exactly $\mathbf{g}(\mathbf{x})$ steps and queries only below $\mathbf{g}(\mathbf{x})$ and outputs a value $\mathbf{y}$ with $\mathbf{y} < \mathbf{g}(\mathbf{x})$. Similarly, if $\varphi_{\mathbf{e}}$ is a Turing reduction and $\varphi_{\mathbf{e}}^{\mathbf{A}}(\mathbf{x})$ needs $\mathbf{s}$ steps to converge then all places queried are below $\mathbf{s}$ and $\varphi_{\mathbf{e}}^{\mathbf{A}}(\mathbf{x}) < \mathbf{s}$. Inparticular, computations $\varphi_{\mathbf{e}}^{\sigma}(\mathbf{x})$ query σ only at values in the domain σ when they terminate and terminate within $|\sigma|$ steps; otherwise they are undefined.

Fact. One can prune of the finite branches of an existing recursive tree and receives a co-r.e. tree in which every node has either one or two successors, but which does not have a deadend. The infinite branches of both are the same.

Main Result

Theorem. There is a co-r.e. tree $\mathbf{S}$ with uncountably many infinite branches such that every nonrecursive infinite branch of $\mathbf{S}$ is of hyperimmune-free Turing degree, minimal Turing degree, generalised low$_\mathbf{1}$, jump traceable and Schnorr trivial. The tree $\mathbf{S}$ is constructed using a $\mathbf{K}$-r.e. guidance tree $\mathbf{T}'$ having only $\mathbf{2}$-generic infinite branches and every infinite branch $\mathbf{A}$ of $\mathbf{S}$ is either isolated and recursive or satisfies that there is a $\mathbf{B}$ with $\mathbf{A} \oplus \mathbf{K} =_\mathbf{T} \mathbf{T}[\mathbf{B}] \oplus \mathbf{K}$ (thus $\mathbf{A} \not\leq_\mathbf{T} \mathbf{K}$) and $\mathbf{A} \oplus \mathbf{K}' =_\mathbf{T} \mathbf{B} \oplus \mathbf{K}'$; furthermore every infinite branch of $\mathbf{A}$ satisfies $\mathbf{A}' =_\mathbf{T} \mathbf{A} \oplus \mathbf{K}$ and $\mathbf{A}'' =_\mathbf{T} \mathbf{A} \oplus \mathbf{K}'$; note that $\mathbf{T}[\mathbf{B}]$ is a $\mathbf{2}$-generic infinite branch of $\mathbf{T}'$ as well.

Construction I

Proof. Construction. The construction works with markers $\mathbf{c_n}$ which are initialised as $\mathbf{c_{n,0} = n}$ and move at stages $\mathbf{t}$ from $\mathbf{c_{n,t}}$ to $\mathbf{c_{n,t+1}}$; in parlallel to the movement of markers, a co-r.e. tree will be constructed which has branching nodes only on levels on which a marker sits. This tree is called $\mathbf{S}$ and $\mathbf{S_0 = \{0,1\}^*}$.

Recall that a finite tree up to a level can be described by the leave nodes in that level; furthermore, one can for each leaf describe the string of branching bits which are those bits in the path to the leaf where one has to make a choice between two possibilities in order to select the leaf to which one wants to go; those where there is only one choice are omitted from the string.

Construction II

At stage $\mathbf{t}$, one first considers for each $\mathbf{n}$ with $\mathbf{c_{n,t} \leq t+1}$ the finite treee $\mathbf{L_n}$ which contains all words $\sigma \in \mathbf{S_t}$ with $\mathbf{dom}(\sigma) \subseteq \{\mathbf{0}, \mathbf{1}, \ldots, \mathbf{c_{n,t}} - \mathbf{1}\}$. One considers an extension of $\mathbf{H_n}$ of $\mathbf{L_n}$ and calls it admissible when it satisfies the following conditions:

1. The leaves of $\mathbf{H_n}$ have the domain $\{\mathbf{0}, \mathbf{1}, \ldots, \mathbf{t}+\mathbf{1}\}$;
2. For each leaf $\sigma \in \mathbf{L_n}$ there is exactly one node τ in $\mathbf{L_n}$ which extends σ and has domain $\{\mathbf{0}, \mathbf{1}, \ldots, \mathbf{t}\}$; furthermore there are either one or two $\mathbf{a} \in \{\mathbf{0}, \mathbf{1}\}$ with $\tau\mathbf{a}$ in $\mathbf{H_n}$;
3. $\mathbf{H_n} \subseteq \mathbf{S_t}$;
4. Let μ is the selection of branching bits of a leaf σ of $\mathbf{L_n}$ and τ for σ as in item $\mathbf{2}$; now if μ is enumerated into $\mathbf{S}$ using oracle $\mathbf{K_t}$ instead of oracle $\mathbf{K}$ within $\mathbf{n}$ steps then $\tau\mathbf{0}, \tau\mathbf{1}$ are both in $\mathbf{H_n}$ else only $\tau\mathbf{0} \in \mathbf{H_n}$;

Construction III

5. Furthermore there is progress from $\mathbf{L_n}$ to $\mathbf{H_n}$ in the way that certain enumeration, definability or splitting goals are obtained; more precisely there are $\sigma, \sigma' \in \mathbf{L_n}$ with the corresponding τ, τ' such that at least one of the following conditions is satisfied:
   (a) Some element $\mathbf{x} \leq \mathbf{n}$ became enumerated into $\mathbf{K}$ between time $\mathbf{c_{n,t}}$ and $\mathbf{t+1}$;
   (b) There is an $\mathbf{e} \leq \mathbf{n}$ for which the function $\varphi^\sigma_\mathbf{e}$ is not defined on all of $\mathbf{0}, \mathbf{1}, \ldots, \mathbf{n}$ within $\mathbf{c_{n,t}}$ steps while $\varphi^\tau_\mathbf{e}$ is defined on all of $\mathbf{0}, \mathbf{1}, \ldots, \mathbf{n}$ within $\mathbf{t+1}$ steps;
   (c) There is an $\mathbf{e} \leq \mathbf{n}$ for which $\varphi^\sigma_\mathbf{e}(\mathbf{e})$ is not defined within $\mathbf{c_{n,t}}$ steps while $\varphi^\tau_\mathbf{e}(\mathbf{e})$ is defined within $\mathbf{t+1}$ steps;

Construction IV

5. (d) For some $\mathbf{e} \leq \mathbf{n}$ there is an $\mathbf{x}$ such that $\varphi_{\mathbf{e}}^{\sigma}$ and $\varphi_{\mathbf{e}}^{\sigma'}$ are consistent at time $\mathbf{c_{n,t}}$ while $\varphi_{\mathbf{e}}^{\tau}(\mathbf{x}), \varphi_{\mathbf{e}}^{\tau'}(\mathbf{x})$ are defined and different for an $\mathbf{x} < \mathbf{t+1}$ within $\mathbf{t+1}$ steps;
   (e) $|\sigma| = |\tau|$ and $\mathbf{c_{n,t} = t+1}$.

For the least $\mathbf{n}$ where one can find $\mathbf{L_n, H_n}$, do:

1. For $\mathbf{m < n}$, $\mathbf{c_{m,t+1} = c_{m,t}}$ and $\mathbf{c_{n,t+1} = t+1}$ and for $\mathbf{m > n}$, $\mathbf{c_{m,t+1} = c_{n,t+1} + n - m}$;
2. Furthermore, one lets $\mathbf{S_{t+1}}$ consist of all the nodes which are comparable to one of the leaves of $\mathbf{H_n}$, that is, one prunes off the tree all nodes which are extending some leaf σ of $\mathbf{L_n}$ but which are incomparable to all leaves of $\mathbf{H_n}$.

Note that such an $\mathbf{n}$ is always found as the $\mathbf{n}$ with $\mathbf{c_{n,t} = t+1}$ can be selected when no smaller $\mathbf{n}$ qualifies.

Notations on $\mathbf{L_n}$ and $\mathbf{H_n}$

Notation. In the following, the final value of $\mathbf{L_n}$ and $\mathbf{H_n}$ refer to the trees with leaves of level $\mathbf{c_{n,t} - 1}$ and $\mathbf{c_{n,t}}$, respectively, where $\mathbf{t}$ is so large that $\mathbf{c_{n,t}}$ does not change at $\mathbf{t}$ or later.

Jump Traceability

Verification of Jump Traceability. One can see by induction that $\mathbf{L_n}$ has always prior to step $\mathbf{t}$ at most $\mathbf{2^n}$ leaves and that $\mathbf{H_n}$ (due to the branching nodes on the level of $\mathbf{c_{n,t+1}}$ being there) can have at most $\mathbf{2^{n+1}}$ leaves. Furthermore, if one enumerates the various possibility for progress in item 5, one sees that the overall number of accounts is bounded by $\mathbf{1 + (n+1) \cdot (1 + 2^n + 2^n + 4^n) \leq 8^{n+2}}$. Now taking into account that activity of smaller $\mathbf{c_m}$ cause $\mathbf{c_n}$ to act again, one gets that $\mathbf{c_n}$ is modified at most $\mathbf{8^{n+2}}$ times between any two changes of lower markers, so in total at most $\mathbf{8^{(n+1)(n+2)}}$ times. Now one can for each leaf τ of each version of $\mathbf{L_n}$ enumerate at most one value of $\varphi^\sigma_{\mathbf{e}}(\mathbf{e})$ and for each infinite branch $\mathbf{A}$ of $\mathbf{S}$ where $\varphi^{\mathbf{A}}_{\mathbf{e}}(\mathbf{e})$ is defined, $\varphi^{\mathbf{A}}_{\mathbf{e}}(\mathbf{e}) = \varphi^\sigma_{\mathbf{e}}(\mathbf{e})$ for one of the leaves of the final version of $\mathbf{L_n}$ after the last time that the marker $\mathbf{c_{n,t}}$ moved, thus every infinite branch $\mathbf{A}$ is jump traceable with bound $\mathbf{2^{3n^2+10n+6}}$.

Tree Property

A Tree Property. An infinite branch $\mathbf{A}$ of $\mathbf{S}$ is isolated iff there is an $\mathbf{n}$ such that the leaf of $\mathbf{A}$ restricted to final version of $\mathbf{L_n}$ has selecting bits μ with $\mu \notin \mathbf{T'}$.

Verification of Tree Property. The reason for this that if μ is not in $\mathbf{T'}$ then for all $\mathbf{m \geq n}$ and the $\mathbf{t}$ where $\mathbf{c_{m,t}}$ sits on its final position and that is at most $\mathbf{t}$, the corresponding $\mathbf{K_t}$ will coincide with $\mathbf{K}$ on all places queried for the first $\mathbf{m}$ steps of the enumeration of $\mathbf{T'}$ and therefore μ will not be enumerated and there will be no branching at the leaf of $\mathbf{A}$ at the level $\mathbf{c_{m,t}}$ and preliminary branchings of this marker are cut off up to this stage. If however for every $\mathbf{n}$ the final $\mathbf{L_n}$ which is the last $\mathbf{H_n}$ without the upmost level after the last revision of $\mathbf{c_{n,t}}$ satisfies that $\mu \in \mathbf{T'}$ then there will be a $\mathbf{m \geq n}$ where μ will actually be enumerated into $\mathbf{T'}$ within $\mathbf{m}$ steps and therefore at that level there will branching node of $\mathbf{S}$ in the final tree $\mathbf{H_n}$ and thus in $\mathbf{S}$. So $\mathbf{A}$ will not be isolated.

Hyperimmune-Free I

Verification of Hyperimmune-Freeness. Let $\mathbf{A}$ be an infinite branch of $\mathbf{S}$ which is not isolated and which therefore might be non-recursive; note that all isolated infinite branches are recursive. Now $\mathbf{A} = \mathbf{S}[\mathbf{B}]$ for some $\mathbf{2}$-generic $\mathbf{B}$. For a given Turing reduction $\varphi_{\mathbf{e}}$, consider the following set:
$\mathbf{W}^{\mathbf{K}} = \{\mu : \exists \mathbf{n} \geq \mathbf{e}$: the final $\mathbf{L_n}$ has a leaf σ with branching bits μ and $\varphi_{\mathbf{e}}^{\sigma}(\mathbf{x})$ being undefined for some $\mathbf{x} < \mathbf{n}\}$. This set is $\mathbf{K}$-r.e. and note that when $\varphi_{\mathbf{e}}^{\mathbf{A}}(\mathbf{x})$ would be defined for some infinite branch $\mathbf{A}$ of $\mathbf{S}$ extending σ, then some progress would be possible and the $\mathbf{L_n}$ would not be the final one.

Hyperimmune-Free II

If an $\mathbf{A} = \mathbf{S}[\mathbf{B}]$ satisfies that there are above every σ some infinite branch $\tilde{\mathbf{A}} = \mathbf{S}[\tilde{\mathbf{B}}]$ of $\mathbf{S}$ also extending σ such that $\varphi_{\mathbf{e}}^{\tilde{\mathbf{A}}}$ is not total, then there is an $\mathbf{x}$ with $\varphi_{\mathbf{e}}^{\tilde{\mathbf{A}}}(\mathbf{x})$ being undefined and any $\mathbf{n} \geq \mathbf{x}$ satisfies that some prefix $\tilde{\sigma}$ of $\tilde{\mathbf{A}}$ is in $\mathbf{L_n}$ as a leaf and $\varphi_{\mathbf{e}}^{\tilde{\sigma}}$ is not defined on all of $\mathbf{0}, \mathbf{1}, \ldots, \mathbf{n}$ and therefore the string ν of its branching bits is in $\mathbf{W^K}$. Thus $\mathbf{B}$ has above every node some extension in $\mathbf{W^K}$ and so $\mathbf{B}$ has a prefix itself which is in $\mathbf{W^K}$. This implies that $\varphi_{\mathbf{e}}^{\mathbf{A}}$ is not total by the definition of $\mathbf{W^K}$.

Hyperimmune-Free III

So one sees that there is a $\sigma \preceq \mathbf{A}$ such that for all infinite branches $\tilde{\mathbf{A}}$ of $\mathbf{S}$ which extend σ it holds that $\varphi_{\mathbf{e}}^{\mathbf{A}}$ is total. Thus one can, for every $\mathbf{x}$, simulate the coenumeration of $\mathbf{S}$ and the pruning of the tree until a stage is found and a level ℓ such that for all $\tau \in \{\mathbf{0}, \mathbf{1}\}^{\ell}$ which extend σ and which are still in $\mathbf{S}$ it holds that $\varphi_{\mathbf{e}}^{\tau}(\mathbf{x})$ converges within ℓ steps and then the maximum of these values is an upper bound for $\varphi_{\mathbf{e}}^{\mathbf{A}}(\mathbf{x})$; as this upper bound is computed by a recursive function, $\varphi_{\mathbf{e}}^{\mathbf{A}}$ has a recursive upper bound. Thus, for each $\mathbf{e}$, either $\varphi_{\mathbf{e}}^{\mathbf{A}}$ is partial (as indicated in the preceding paragraph) or $\varphi_{\mathbf{e}}^{\mathbf{A}}$ has a recursive upper bound and therefore all non-isolated infinite branch of $\mathbf{S}$ is of hyperimmune-free Turing degree. The isolated infinite branches are recursive and therefore of hyperimmune-free Turing degree as well.

Schnorr Trivial

Verification of Schnorr trivial. If φ_e is a truth-table reduction with bound function g for the computation time then it will happen for almost all n that $c_{n,t}$ is eventually above $g(n)$; this is due to the fact that the time to enumerate the halting time of K up to n is a dominating function with respect to n. Thus one can, for all sufficiently large n, simulate the construction until the final value of the $c_{n,t}$ is above the use and then enumerate the 2^n values which are defined by the various branches of the co-r.e. tree which survive. The finitely many smaller values can be patched. Thus A is Schnorr-trivial by a characterisation of Franklin and Stephan [Schnorr-trivial sets and truth-table reducibility, JSL 75(2):501-521, 2010].

Double Jump of Infinite Branches

Double-Jumps of the infinite branches are exactly the cones above $\mathbf{K}'$. Assume $\mathbf{A}$ is a non-isolated infinite branch of $\mathbf{S}$; then $\mathbf{A} = \mathbf{S}[\mathbf{B}]$ for some infinite branch $\mathbf{B}$ of $\mathbf{T}$. Furthermore, $\mathbf{K}'$ allows to reconstruct all branching nodes of $\mathbf{T}$ and the branching bits of $\mathbf{B}$ with respect to $\mathbf{T}$ give a set $\mathbf{F}(\mathbf{B})$ which can be computed from $\mathbf{A} \oplus \mathbf{K}'$. It follows that the mapping $\mathbf{A} \mapsto \mathbf{deg_T}(\mathbf{A} \oplus \mathbf{K}')$ is surjection from the infinite branches of $\mathbf{S}$ to the Turing degrees above $\mathbf{K}'$. Furthermore, $\mathbf{A}' =_\mathbf{T} \mathbf{A} \oplus \mathbf{K}$ due to $\mathbf{A}$ being jump traceable and $\mathbf{A}'' =_\mathbf{T} (\mathbf{A} \oplus \mathbf{K})' =_\mathbf{T} \mathbf{T}[\mathbf{B}] \oplus \mathbf{K}'$ by $\mathbf{T}[\mathbf{B}]$ being $\mathbf{2}$-generic. Furthermore, Marcus Triplett mentions in his bachelor thesis (Theorem 4.26, https://www.cs.auckland.ac.nz/˜nies/Students/TriplettKTriviality.pdf) that every HIF set $\mathbf{S}$ satisfies $\mathbf{A}'' =_\mathbf{T} \mathbf{A} \oplus \mathbf{K}'$. This result was also obtained by Martin before.

Minimal Turing Degrees I

Every nonrecursive infinite branch $\mathbf{A}$ of $\mathbf{S}$ has minimal Turing degree. Let $\mathbf{A}$ be a non-recursive infinite branch of $\mathbf{S}$. Consider a total function $\varphi_{\mathbf{e}}^{\mathbf{A}}$. Then there is a σ such that all infinite branches $\tilde{\mathbf{A}}$ of $\mathbf{S}$ above σ, $\varphi_{\mathbf{e}}^{\tilde{\mathbf{A}}}$ is total.

If now there is such an infinite branch $\tilde{\mathbf{A}}$ extending σ which is different from $\mathbf{A}$ but for which $\varphi_{\mathbf{e}}^{\mathbf{A}}$ and $\varphi_{\mathbf{e}}^{\tilde{\mathbf{A}}}$ coincide, then it follows that above the branching node σ' of $\mathbf{A}$ and $\tilde{\mathbf{A}}$, all infinite branches of $\mathbf{S}$ produce the same function $\varphi_{\mathbf{e}}^{\mathbf{A}}$ and so this function is recursive; otherwise $\sigma'\mathbf{0}$ or $\sigma'\mathbf{1}$ would be extended into a sufficiently long prefixes of $\hat{\mathbf{A}}$ and the respective of $\mathbf{A}$ and $\tilde{\mathbf{A}}$ in order to achieve that these prefixes are mapped by $\varphi_{\mathbf{e}}$ on some value to different images and so either $\mathbf{A}$ or $\tilde{\mathbf{A}}$ would be cut off.

Minimal Turing Degrees II

So one has that there is a $\sigma'' \preceq \mathbf{A}$ such that, for all infinite branches $\tilde{\mathbf{A}}$ of $\mathbf{S}$ extending σ'', either the functions $\varphi_{\mathbf{e}}^{\tilde{\mathbf{A}}}$ are all different or are all the same.

If they are all different, one can Turing reduce $\mathbf{A}$ to $\varphi_{\mathbf{e}}^{\mathbf{A}}$. Whenever there are two different extensions above some $\sigma''' \succeq \sigma''$, one coenumerates $\mathbf{S}$ and simulates $\varphi_{\mathbf{e}}$ until either an $\mathbf{e}$-splitting at some $\mathbf{x}$ above $\sigma'''\mathbf{0}$ and $\sigma'''\mathbf{1}$ is found so that $\varphi_{\mathbf{e}}^{\mathbf{A}}(\mathbf{x})$ says which branch to follow or one of the two nodes $\sigma'''\mathbf{0}$ and $\sigma'''\mathbf{1}$ has been enumerated out of $\mathbf{S}$. Thus $\mathbf{A} \equiv_{\mathbf{T}} \varphi_{\mathbf{e}}^{\mathbf{A}}$ by the usual $\mathbf{e}$-splitting analysis.

If the infinite branches of $\mathbf{S}$ above σ'' have under $\varphi_{\mathbf{e}}$ all the same image, then one can for each input $\mathbf{x}$ coenumerate $\mathbf{S}$ until a level $\ell > |\sigma''|$ and a time $\mathbf{t}$ are found such that all $\tau \in \{\mathbf{0},\mathbf{1}\}^{\ell} \cap \mathbf{S}_{\mathbf{t}}$ which extend σ'' satisfy that $\varphi_{\mathbf{e},\mathbf{t}}^{\tau}(\mathbf{x})$ is a unique value $\mathbf{y}$ and equals to $\varphi_{\mathbf{e}}^{\mathbf{A}}(\mathbf{x})$. Thus minimality of $\mathbf{A}$ is verified and the proof is completed.

Perfectness of Nonisolated Branches

There is a $\mathbf{K}$-recursive operator $\mathbf{F}$ which maps the infinite branch $\mathbf{T}[\mathbf{B}]$ of $\mathbf{T}$ to the corresponding infinite branch $\mathbf{A}$ of $\mathbf{S}$; this mapping is one-one and continuous. Furthermore, if $\mathbf{B_0}, \mathbf{B_1}, \ldots$ converge pointwise against $\mathbf{B}$ and all $\mathbf{B_k} \neq \mathbf{B}$ then the infinite branches $\mathbf{T}[\mathbf{B_k}]$ converge pointwise to $\mathbf{T}[\mathbf{B}]$ in $\mathbf{T}$ and $\mathbf{F}(\mathbf{T}[\mathbf{B_k}])$ converge pointwise to $\mathbf{F}(\mathbf{T}[\mathbf{B}])$. The mapping $\mathbf{B} \mapsto \mathbf{F}(\mathbf{T}[\mathbf{B}])$ is one-one and continuous. Hence the nonrecursive infinite branches of $\mathbf{S}$ coincide with the nonisolated infinite branches and form a perfect subtree of $\mathbf{S}$ which is co-r.e. relative to $\mathbf{K}$.

Remark

Remark. Instead of taking $\mathbf{T}$ to be the union of all $\mathbf{T}[\mathbf{B}]$ when making the result, one could take a $\mathbf{K}'$-recursive tree with uncountably many $\mathbf{K}'$-hyperimmune-free and $\mathbf{2}$-generic branches and then get the result that there is a tree $\mathbf{S}$ with uncountably many infinite branches such that every infinite branch is hyperimmune-free, hyperimmune-free relative to $\mathbf{K}'$, jump-traceable and jump-traceable relative to $\mathbf{K}'$. It is possible to iterate this.

Summary. This talk gave the proof of the following theorem.

Theorem. There is a co-r.e. tree $\mathbf{T}$ without finite branches having $\mathbf{2}^{\aleph_0}$ many infinite branches such that all infinite branches are hyperimmune-free, generalised low, Schnorr trivial, of minimal Turing degree and jump traceable.