Homework for 26.08.2004

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Homework. The homework follows the lecture notes. What cannot be done as scheduled, will be done the week afterwards. Lecture is Mon 16.00h - 17.30h and Thu 16.00h - 16.45h. Tutorial is Thu 16.45h - 17.30h. The room is S13#05-03.

http://www.comp.nus.edu.sg/~fstephan/homework.ps
http://www.comp.nus.edu.sg/~fstephan/homework.pdf

Please sit as near to Whiteboard as possible. There is not much space on the whiteboard, thus the larger the writing the less can be put there. Try to sit near in order to be able to read the writing.

Exercise 1.5. The property of being well-founded is an abstract property which applies also to some but not all graphs which are different from the universe of all sets. Here some examples of graphs. Which of them are well-founded? The answers should be proven.

1. the set $\{0, 1, \{0\}, \{1\}, \{0, 1, \{0\}\}, \{\{1\}\}, \{\{\{1\}\}\}, 512\}$ with (a, b) being an edge iff $a \in b$;
2. the set $\{0, 1, 2, 3\}$ with the edges $(0, 1), (1, 0), (2, 3)$;
3. the set $\mathbb{N}$ of the natural numbers with every edge being of the form $(n, n + 1)$;
4. the set $\mathbb{Z}$ of the integers with the edges being the pairs $(n, n + 1)$ for all $n \in \mathbb{Z}$;
5. the set $\mathbb{Q}$ of rational numbers with the edges being the pairs $(q, 2q)$ for all $q \in \mathbb{Q}$;
6. the set $\mathbb{Q}$ of rational numbers with the edges being the pairs $(q, q + 1)$ for all $q \geq 0$ and $(q, q - 1)$ for all $q \leq 0$.

Exercise 1.12. Which of the following sets of natural numbers are equal? Well-known mathematical theorems can be applied without proving them.

1. $A = \{1, 2\}$;
2. $B = \{1, 2, 3\}$;
3. C is the set of all prime numbers;
4. $D = \{d \mid \exists a, b, c > 0\, (a^d + b^d = c^d)\}$;
5. $E = \{e \mid e > 0 \wedge \forall c \in C\, (e \leq c)\}$;
6. $F = \{f \mid \forall c \in C\, (f \geq c)\}$;
7. $G = \{g \mid g \geq 2 \wedge \forall a, b > 1\, (4g \neq (a+b)^2 - (a-b)^2)\}$;
8. $H = \{h \mid h > 0 \wedge h^2 = h^h\}$;
9. $I = \{i \mid i + i = i \cdot i\}$;
10. $J = \{j \mid (j+1)^2 = j^2 + 2j + 1\}$;
11. $K = \{k \mid 4k > k^2\}$;
12. $L = \{l \mid \exists c \in C\, (l < c)\}$;
13. $M = \{m \mid \exists c \in C\, (m = c^2)\}$;
14. $N = \{n \mid \exists c, d \in C\, (n = cd)\}$;
15. $O = \{o \mid o \text{ has exactly three prime factors}\}$;
16. $P = \{p \mid p, p + 2 \in C\}$.

Exercise 2.5. Determine the power-set of $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$. Is there any set X such that $\mathcal{P}(X)$ has exactly 9 elements?

Exercise 2.9. Establish properties to define the following sets as subsets of the natural numbers using the Axiom of Comprehension:

1. the set of all numbers with exactly three divisors,
2. the set $\{0, 2, 4, 6, \ldots\}$ of all even numbers,
3. the set of all square numbers,
4. the set of all numbers whose binary representation contains exactly four times a 1.

For example, the set of prime numbers can be defined as the set $\{x \in \mathbb{N} \mid \exists$ unique $y, z \in \mathbb{N}\,(y \cdot z = x \wedge z < y \leq x)\}$, that is the set of all natural numbers with exactly two divisors.

Exercise 2.10. Show that every property p satisfies the following statements.

1. There are sets x, y such that $x \in y$ and either $p(x) \wedge p(y)$ or $\neg p(x) \wedge \neg p(y)$.
2. There is a set x with $x = \{y \in x \mid p(y)\}$.
3. There is a one-to-one function f such that $p(x)$ iff $p(y)$ for all $y \in f(x)$.

Exercise 2.18. Prove that the symmetric difference is associative, that is, for all sets A, B, C, $(A \triangle B) \triangle C = A \triangle (B \triangle C)$. For this reason, one can just write $A \triangle B \triangle C$. Furthermore, prove that $A - B = A \cap (A \triangle B)$.

Exercise 2.19. Consider the sets *Apple*, *Pear*, *Strawberry*, *Cranberry*, *Blackberry*, *Banana*, *Blueberry* which consist of all fruits in the world usually designated by that name. Let *Fruits* be the union of these sets and *Red*, *Blue*, *Black* and *Yellow* be those elements of *Fruits* which have the corresponding colour. Which of the following expressions is the empty set?

1. $Apple - Red$,
2. $(Black \triangle Blueberry) \cap Blue$,
3. $Fruit - Red - Blue - Black - Yellow$,
4. $Red - Strawberry - Cranberry - Apple - Pear$,
5. $(Blueberry - Blue) \cup (Yellow - Apple - Pear - Banana)$,
6. $Banana - Yellow$,
7. $Banana \triangle Blueberry \triangle Strawberry \triangle Red$,
8. $(Strawberry \cup Blueberry \cup Cranberry) - Red$,
9. $(Apple \cup Pear) \cap (Strawberry \cup Blueberry)$,
10. $Fruit - \bigcup\{Apple, Pear, Strawberry, Cranberry, Blackberry, Banana\}$.

Give a set of three fruits which intersects those of the above sets which are not empty.

Exercise 2.22. Many Boolean Algebras have a complementation operation, but in Property 2.21 only the set difference is used in the De Morgan Laws below for sets A, B, C. Why?

$$\text{De Morgan Laws:} \quad \begin{array}{rcl} A - (B \cap C) & = & (A - B) \cup (A - C), \\ A - (B \cup C) & = & (A - B) \cap (A - C). \end{array}$$