Homework for 02.09.2004

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Homework. The homework follows the lecture notes. What cannot be done as scheduled, will be done the week afterwards. This homework contains the left-overs from 26.08.2004.

Lecture is Mon 16.00h - 17.30h and Thu 16.00h - 16.45h. Tutorial is Thu 16.45h - 17.30h. The room is S13#05-03.

http://www.comp.nus.edu.sg/~fstephan/homework.ps
http://www.comp.nus.edu.sg/~fstephan/homework.pdf

Exercise 1.12. Which of the following sets of natural numbers are equal? Well-known mathematical theorems can be applied without proving them.

1. $A = \{1, 2\}$;
2. $B = \{1, 2, 3\}$;
3. C is the set of all prime numbers;
4. $D = \{d \mid \exists a, b, c > 0\,(a^d + b^d = c^d)\}$;
5. $E = \{e \mid e > 0 \wedge \forall c \in C\,(e \leq c)\}$;
6. $F = \{f \mid \forall c \in C\,(f \geq c)\}$;
7. $G = \{g \mid g \geq 2 \wedge \forall a, b > 1\,(4g \neq (a + b)^2 - (a - b)^2)\}$;
8. $H = \{h \mid h > 0 \wedge h^2 = h^h\}$;
9. $I = \{i \mid i + i = i \cdot i\}$;
10. $J = \{j \mid (j + 1)^2 = j^2 + 2j + 1\}$;
11. $K = \{k \mid 4k > k^2\}$;
12. $L = \{l \mid \exists c \in C\,(l < c)\}$;

13. $M = \{m \mid \exists c \in C\,(m = c^2)\}$;
14. $N = \{n \mid \exists c, d \in C\,(n = cd)\}$;
15. $O = \{o \mid o \text{ has exactly three prime factors}\}$;
16. $P = \{p \mid p, p+2 \in C\}$.

Exercise 2.9. Establish properties to define the following sets as subsets of the natural numbers using the Axiom of Comprehension:

1. the set of all numbers with exactly three divisors,
2. the set $\{0, 2, 4, 6, \ldots\}$ of all even numbers,
3. the set of all square numbers,
4. the set of all numbers whose binary representation contains exactly four times a 1.

For example, the set of prime numbers can be defined as the set $\{x \in \mathbb{N} \mid \exists \text{ unique } y, z \in \mathbb{N}\,(y \cdot z = x \wedge z < y \leq x)\}$, that is the set of all natural numbers with exactly two divisors.

Exercise 2.10. Show that every property p satisfies the following statements.

1. There are sets x, y such that $x \in y$ and either $p(x) \wedge p(y)$ or $\neg p(x) \wedge \neg p(y)$.
2. There is a set x with $x = \{y \in x \mid p(y)\}$.
3. There is a one-to-one function f such that $p(x)$ iff $p(y)$ for all $y \in f(x)$.

Exercise 2.18. Prove that the symmetric difference is associative, that is, for all sets A, B, C, $(A \triangle B) \triangle C = A \triangle (B \triangle C)$. For this reason, one can just write $A \triangle B \triangle C$. Furthermore, prove that $A - B = A \cap (A \triangle B)$.

Exercise 3.9. Define (informally) functions f_n from $\mathbb{N}$ to $\mathbb{N}$ with the following properties:

1. f_1 is bijective and satisfies $f_1(x) \neq x$ but $f_1(f_1(x)) = x$ for all $x \in \mathbb{N}$;
2. f_2 is two-to-one: for every y there are exactly two elements $x, x' \in \mathbb{N}$ with $f(x) = f(x') = y$;
3. f_3 is dominating all polynomials, that is, for every polynomial p there is an x such that for all $y > x$, $f_3(y) > p(y)$;

4. f_4 satisfies $f_4(x+1) = f_4(x) + 2x + 1$ for all $x \in \mathbb{N}$;

5. $f_5(x) = \begin{cases} 0 & \text{if } x = 0; \\ f_5(x-1) & \text{if } x > 0 \text{ and } x \text{ is not a square number;} \\ f_5(x-1) + 1 & \text{if } x > 0 \text{ and } x \text{ is a square number.} \end{cases}$

Determine the range of the function f_4.

Exercise 3.11. Let $A = \{0, 1, 2\}$ and $F = \{f : A \to A \mid f = f \circ f\} = \{f : A \to A \mid \forall x\,(f(x) = f(f(x)))\}$. Show that F has exactly 10 members and determine these.