Homework for 07.10.2004

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Homework. The homework follows the lecture notes. What cannot be done as scheduled, will be done the week afterwards. This homework contains the left-overs from 30.09.2004.

In general, lecture is Mon 16.00h - 17.30h and Thu 16.00h - 16.45h. Tutorial is Thu 16.45h - 17.30h. On 14.10.2004 there are 90min of lecture. The room is S13#05-03.

http://www.comp.nus.edu.sg/˜fstephan/homework.ps
http://www.comp.nus.edu.sg/˜fstephan/homework.pdf

Exercise 9.6. Let $A = \mathbb{N} - \{0, 1\} = \{2, 3, 4, \ldots\}$ and let $<_{div}$ be given by $x <_{div} y \Leftrightarrow \exists z \in A\,(x \cdot z = y)$. That is, $x <_{div} y$ iff x is a proper divisor of y, so $2 <_{div} 8$ but $2 \not<_{div} 2$ and $2 \not<_{div} 5$. Prove that $(A, <_{div})$ is a partially ordered set.

Exercise 9.9. Prove that the following relations are partial orderings on $\mathbb{N}^{\mathbb{N}}$:

- $f \sqsubset_1 g \Leftrightarrow \exists n\, \forall m > n\, (f(m) < g(m))$;
- $f \sqsubset_2 g \Leftrightarrow \forall n\, (f(n) \leq g(n)) \wedge \exists m\, (f(m) < g(m))$;
- $f \sqsubset_3 g \Leftrightarrow \forall n\, (f(n) \leq g(n)) \wedge \exists n\, (f(n) < g(n)) \wedge \exists n\, \forall m > n\, (f(m) = g(m))$;
- $f \sqsubset_4 g \Leftrightarrow f(0) < g(0)$.

Determine for every ordering a pair of incomparable elements f, g such that neither $f \sqsubset_m g$ nor $g \sqsubset_m f$ nor $f = g$. For which of these orderings is it possible to choose the f of this pair (f, g) of examples such that $f(n) = 0$ for all n?

Exercise 9.13. Let $A = \mathbb{N} - \{0, 1\}$ and $<_{div}$ given by $x <_{div} y \Leftrightarrow \exists z \in A\,(x \cdot z = y)$ as in Exercise 9.6. Define a relation E on $A \times A$ by putting (x, y) into E iff there is a prime number z with $x \cdot z = y$. So $(2, 4) \in E$, $(2, 6) \in E$, $(2, 10) \in E$ but $(2, 7) \notin E$, $(2, 8) \notin E$ and $(2, 20) \notin E$. Show that $(A, <_E)$ and $(A, <_{div})$ are identical partially ordered sets.

Exercise 10.6. Let $(A, <)$ be a linearly ordered set and $B = A^{\mathbb{N}}$. Define

$$f <_{lex} g \Leftrightarrow (\exists k \in \mathbb{N})\,[f\restriction k = g\restriction k \wedge f(k) < g(k)]$$

Furthermore, let $C = A^*$. The lexicographic ordering on A^* is defined such that either the smaller word is shorter than the longer one or that the first word has a member of A strictly before the second one at the first position where they differ. That is, if m is the domain of f and n the domain of g, then

$$f <_{lex} g \Leftrightarrow \exists k \in S(m) \cap n \, ((f\restriction k = g\restriction k) \wedge (k = m \vee (k < m \wedge f(k) < g(k)))).$$

Show that $(B, <_{lex})$ and $(C, <_{lex})$ are linearly ordered sets. Assuming that $A = \{0, 1, 2, \ldots, 9\}$ with the usual ordering, put the following elements of C into lexicographic order: 120, 88, 512, 500, 5, 121, 900, 0, 76543210, 15, 7, 007, 00.

Exercise 10.10. Determine which of the following subsets of the real numbers $\mathbb{R}$ have a lower and upper bound. If so, determine the infimum and supremum and check whether these are even the least and greatest element of these sets.

1. $A = \{a \in \mathbb{R} \mid \exists b \in \mathbb{R} \, (a^2 + b^2 = 1)\}$;
2. $B = \{b \in \mathbb{R} \mid b^3 - 4 \cdot b < 0\}$;
3. $C = \{c \in \mathbb{R} \mid \sin(c) > 0\}$;
4. $D = \{d \in \mathbb{R} \mid d^2 < \pi^3\}$;
5. $E = \{e \in \mathbb{R} \mid \sin(\frac{\pi}{2} \cdot e) = \frac{e}{101}\}$.

Exercise 10.13. Consider the ordering $\sqsubset$ given by

$$\begin{aligned} (m, n) \sqsubset (i, j) \quad &\Leftrightarrow \quad (m < i) \\ &\vee \quad (m = i \wedge m \text{ is even} \wedge n < j) \\ &\vee \quad (m = i \wedge m \text{ is odd} \wedge n > j) \end{aligned}$$

on $A = \{0, 1, 2, 3, 4, 5\} \times \mathbb{N}$. Construct an order-preserving mapping from $(\mathbb{Z}, <)$ into $(A, \sqsubset)$ where $<$ is the natural ordering of $\mathbb{Z}$.

The set $(\mathbb{Z}, <)$ there are nontrivial isomorphisms onto itself, that is, isomorphism different from the identity. For example, $z \mapsto z + 8$. Does $(A, \sqsubset)$ also have nontrivial isomorphisms onto itself? If so, is there any element which is always mapped to itself?

Exercise 10.22. Show that in a complete ordered set $(A, <)$ every nonempty subset which is bounded from below has an infimum in A.