Homework for 04.11.2004

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Homework. The homework follows the lecture notes. What cannot be done as scheduled, will be done the week afterwards.

Lecture is Mon 16.00h - 17.30h and Thu 16.00h - 16.45h. Tutorial is Thu 16.45h - 17.30h. The room is S13#05-03.

http://www.comp.nus.edu.sg/˜fstephan/homework.ps
http://www.comp.nus.edu.sg/˜fstephan/homework.pdf

Exercise 14.6. Show the following properties of $\oplus$.

1. $\oplus$ is commutative.
2. There are ordinals α, β such that $\alpha + \beta$ and $\beta + \alpha$ both differ from $\alpha \oplus \beta$.
3. There are ordinals α, β such that $\alpha < \beta$ and $\alpha \oplus \gamma \neq \beta$ for all ordinals γ.
4. In contrast to this, there is for each ordinals α, β with $\alpha < \beta$ an ordinal γ with $\alpha + \gamma = \beta$.
5. For all ordinals α, β, $\alpha + \beta \leq \alpha \oplus \beta$.

Exercise 14.10. Determine the Cantor Normal Form of the following ordinals.

1. $\omega + \omega^2 + \omega^3 + \omega^4 + 2$,
2. $(\omega + 3)^5 + (\omega^2 + 17) \cdot (\omega + 8) + \omega^{12}$,
3. $\omega^2 + \omega + 1 + \omega^2 + \omega + 1 + \omega^2 + \omega + 1$,
4. $1 \oplus \omega \oplus \omega^2 \oplus \omega^3$,
5. $\omega^{\omega+5} + \omega^{\omega+2} \cdot \omega + \omega^2$,
6. $256^{256} + \omega \cdot 42$.

Exercise 14.11. Use the Cantor Normal Form to prove the following: Let α, β be ordinals such that $\gamma \geq \delta$ whenever $F_\alpha(\gamma) > 0 \wedge F_\beta(\delta) > 0$. Then $\alpha + \beta = \alpha \oplus \beta$.

Exercise 15.9. Construct a one-to-one function h which maps $\alpha \times \omega$ to α for any infinite limit ordinal α. This function can without loss of generality assume that the input is of the form $(\gamma \cdot \omega + n, m)$ where $m, n \in \mathbb{N}$ and γ is an ordinal with $S(\gamma) \cdot \omega \leq \alpha$; the image should be of the form $\gamma \cdot \omega + \tilde{h}(n, m)$ for some function $\tilde{h} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.