MA 3219 – Computability Theory

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Assignment for 16.03.2005. Can be corrected on request, it is not obligatory to hand the homework in.

1. The Halting Problem. The set $H = \{(e, x) : \phi_e(x) \text{ is defined}\}$ is called the General Halting Problem and is known to be undecidable. Show that for every total and computable function f the f-diagonal problem

$$K_f = \{e : \phi_e(f(e)) \text{ is defined}\}$$

is also undecidable. Use the S-m-n Theorem to show that there is a function g such that

$$(e, x) \in H \Leftrightarrow g(e, x) \in K_f$$

and conclude then that K_f is also undecidable.

2. Numberings. A function ψ is called a numbering for a class F of computable functions iff the following two conditions hold:

- for all $\theta \in F$ there is an index e with $\forall x\, (\psi(e, x) = \theta(x))$;
- for all e there is a $\theta \in F$ such that $\forall x\, (\theta(x) = \psi(e, x))$.

(a) Show that there is a numbering ψ of all functions which are either total or defined on a set of the form $\{y : y < x\}$ for some x. Construct ψ from the universal function ψ_U.

(b) Show that the class of all total computable functions does not have a numbering.

(c) Construct a further class of functions which does not have a numbering.

3. Universal Functions and Other Numberings. A numbering ψ for F is called a universal function for F iff one can translate every further numbering θ of any subset of F into ψ as follows:

$$\exists \text{ total and computable } g\, \forall e\, \forall x\, (\theta(e, x) = \psi(g(e), x)).$$

That is, g computes the number $g(e)$ which the e-th function with respect to θ has with respect to ψ.

Friedberg showed that there is a numbering ψ of all computable functions such that for every different e, e' there is an x such that either $\psi(e, x), \psi(e', x)$ are both defined and different or exactly one of them is defined. That is, the e-th and the e'-th functions are different. Show that Friedberg's numbering is not a universal function.