MA 3205 – Set Theory – Homework for Week 4

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Homework. The homework follows the lecture notes. Below the list of the homeworks for the tutorials from 05.09.2006 onwards. It is not mandatory to hand in homework; but it is recommended to solve the questions by yourself before going to the tutorials. Homework can be corrected on request.

Exercise 3.10. Define (informally) functions f_n from $\mathbb{N}$ to $\mathbb{N}$ with the following properties:

1. f_1 is bijective and satisfies $f_1(x) \neq x$ but $f_1(f_1(x)) = x$ for all $x \in \mathbb{N}$;
2. f_2 is two-to-one: for every y there are exactly two elements $x, x' \in \mathbb{N}$ with $f(x) = f(x') = y$;
3. f_3 is dominating all polynomials, that is, for every polynomial p there is an x such that for all $y > x$, $f_3(y) > p(y)$;
4. f_4 satisfies $f_4(x+1) = f_4(x) + 2x + 1$ for all $x \in \mathbb{N}$;
5. $f_5(x) = \begin{cases} 0 & \text{if } x = 0; \\ f_5(x-1) & \text{if } x > 0 \text{ and } x \text{ is not a square number;} \\ f_5(x-1)+1 & \text{if } x > 0 \text{ and } x \text{ is a square number.} \end{cases}$

Determine the range of the function f_4.

Exercise 3.12. Let $A = \{0, 1, 2\}$ and $F = \{f : A \to A \mid f = f \circ f\} = \{f : A \to A \mid \forall x\, (f(x) = f(f(x)))\}$. Show that F has exactly 10 members and determine these.

Exercise 4.6. Which of the following sets is transitive and which is inductive?

1. $A = \{\emptyset, \{\emptyset\}\}$,
2. $B = \{\emptyset, \{\{\{\emptyset\}\}\}\}$,
3. $C = \{x \mid \forall y \in x\, \forall z \in y\, (z = \emptyset)\}$,
4. D is the closure of $\{\emptyset, \mathbb{N} \times \mathbb{N}\}$ under the successor operation $x \mapsto S(x)$,
5. E is the set of even numbers,
6. F is the set of all natural numbers which can be written down with at most 256 decimal digits,

7. G is the set of all finite subsets of $\mathbb{N}$,

8. $H = \mathcal{P}(G)$.

Exercise 4.7. Show that the following statements are equivalent for any inductive set X.

1. $X = \mathbb{N}$;

2. X has no proper inductive subset;

3. X is a subset of every inductive set;

4. $\forall x \in X\, (x = 0 \vee \exists y \in X\, (x = S(y)))$;

5. $X = N(Y)$ for every inductive set Y where $N(Y)$ is the subset of those $y \in Y$ which are in every inductive subset of Y;

6. $X = N(Y)$ for some inductive set Y where $N(Y)$ is defined as in the previous item.

Exercise 4.9. Assume that a property p satisfies

$$p(1) \text{ and } \forall x\, (p(x) \Rightarrow p(S(S(x)))).$$

Consider the following subsets of natural numbers:

1. the set of all numbers;

2. the set of even numbers;

3. the set of odd numbers;

4. the set of square numbers;

5. the set of all powers of 35, that is, $\{1, 35, 1225, 42875, 1500625, \ldots\}$.

For which of these sets it is guaranteed that all elements and for which it is guaranteed that some elements satisfy p? Consider the property

$$q(x) \Leftrightarrow \forall A \subseteq \mathbb{N}\, ((1 \in A \wedge \forall y \in \mathbb{N}\, (y \in A \Rightarrow S(S(y)) \in A)) \Rightarrow x \in A).$$

Which of the above sets is equal to $\{x \in \mathbb{N} \mid q(x)\}$. That is, which set is the intersection of all sets A for which

$$1 \in A \wedge \forall y \in \mathbb{N}\, (y \in A \Rightarrow S(S(y)) \in A)$$

is true?