MA 3205 – Set Theory – Homework for Week 11

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Homework. The homework follows the lecture notes. Below the list of the homeworks for the tutorials from 31.10.2006 onwards. It is not mandatory to hand in homework; but it is recommended to solve the questions by yourself before going to the tutorials. Homework can be corrected on request.

Rank. The rank is a measure of how much a set is nested, of if there are elements of elements of elements in the set, then its rank is at least 3. Here some easy rules:

- The rank of an ordinal α is α. So $\rho(0) = 0, \rho(1) = 1, \rho(\omega) = \omega$.
- The rank of sets is determined by the rank of their elements. For example, $\rho(\{x, y, z\}) = \max\{\rho(x)+1, \rho(y)+1, \rho(z)+1\}$ and $\rho(\{\omega, \omega\cdot 2, \omega+5\}) = \omega\cdot 2+1$.
- In general, $\rho(X) = \sup\{\rho(Y) + 1 \mid Y \in X\}$.

The Cantor Normal Form. The Cantor Normal Form is a finite sum of powers of ω in decending order. Here an example:

$$\omega^5 + \omega^2 + \omega^2 + \omega^2 + \omega^1 + \omega^0 + \omega^0 \; = \; \omega^5 + \omega^2 \cdot 3 + \omega + 2.$$

Instead of repeating same ordinals, one can also multiply them with the corresponding natural number, instead of ω^1, one can write just ω, instead of ω^0 just 1. This all is done on the right hand side of the equation. Also transfinite ordinals can be in the power:

$$\omega^{\omega^2} + \omega^{\omega\cdot 5+8} \cdot 7 + \omega^{\omega\cdot 5+7} \cdot 12345 + \omega^{22222} \cdot 33333 + \omega^4 + \omega^3 + \omega^2 + \omega + 1.$$

If one adds ordinals $\omega^\alpha \cdot a + \omega^\beta \cdot b$ with $\alpha < \beta$, then $\omega^\alpha \cdot a$ can be omitted; if $\alpha = \beta$ the coefficients can be added giving $\omega^\alpha \cdot (a+b)$; if $\alpha > \beta$, no simplification is possible:

$$\begin{aligned}
\omega^3 + \omega^5 &= \omega^5; \\
\omega^3 \cdot 5 + \omega^4 \cdot 8 + \omega^5 \cdot 0 &= \omega^4 \cdot 8; \\
\omega^3 \cdot 234 + \omega^3 \cdot 111 &= \omega^3 \cdot 345; \\
\omega^5 + \omega^3 + \omega^2 + \omega^3 + \omega^1 &= \omega^5 + \omega^3 \cdot 2 + \omega.
\end{aligned}$$

The last line has the application of two rules: first ω^2 is omitted as it is in front of a higher ω-power; second the two ω^3-terms are unified to one; no further simplification is possible.

Exercise 14.4. For any ordinal α, consider the successor function S restricted to α, that is, consider the set

$$S\restriction\alpha = \{\{\beta, \{\beta, S(\beta)\}\} \mid \beta \in \alpha\}.$$

Determine $\rho(S\restriction\alpha)$ for $\alpha = 42, 1905, 2004, \omega, \omega+1, \omega+131501, \omega^2+\omega\cdot 2+1, \omega^{17}+\omega^4$.

Exercise 14.6. V_ω has been defined twice. Let A be the version of V_ω as defined in Definition 7.5, that is, let A consist of all hereditarily finite sets. Let $B = \bigcup\{V_n \mid n < \omega\} = \{x \in V \mid \rho(x) < \omega\}$ be the version defined here. Show that both definitions coincide, that is, show $A \subseteq B \wedge B \subseteq A$.

Show that B contains $\emptyset$, is closed under unions of two sets and is closed under the operation forming $\{v\}$ from v. Thus, by Theorem 7.9, $A \subseteq B$.

Show by induction that all members of V_n with $n < \omega$ are hereditarily finite. Thus $B \subseteq A$.

Exercise 15.6. Show the following properties of $\oplus$.

1. $\oplus$ is commutative.
2. There are ordinals α, β such that $\alpha+\beta$ and $\beta+\alpha$ both differ from $\alpha\oplus\beta$.
3. There are ordinals α, β such that $\alpha < \beta$ and $\alpha \oplus \gamma \neq \beta$ for all ordinals γ.
4. In contrast to this, there is for each ordinals α, β with $\alpha < \beta$ an ordinal γ with $\alpha + \gamma = \beta$.
5. For all ordinals α, β, $\alpha + \beta \leq \alpha \oplus \beta$.

Exercise 15.10. Determine the Cantor Normal Form of the following ordinals.

1. $\omega + \omega^2 + \omega^3 + \omega^4 + 2$,
2. $(\omega+3)^5 + (\omega^2+17)\cdot(\omega+8) + \omega^{12}$,
3. $\omega^2+\omega+1+\omega^2+\omega+1+\omega^2+\omega+1$,
4. $1 \oplus \omega \oplus \omega^2 \oplus \omega^3$,
5. $\omega^{\omega+5} + \omega^{\omega+2}\cdot\omega + \omega^2$,
6. $256^{256} + \omega\cdot 42$.

Exercise 15.11. Use the Cantor Normal Form to prove the following: Let α, β be ordinals such that $\gamma \geq \delta$ whenever $F_\alpha(\gamma) > 0 \wedge F_\beta(\delta) > 0$. In other words, $\gamma \geq \delta$ whenever the ω-power ω^γ occurs in the Cantor Normal Form of α and the ω-power ω^δ occurs in the Cantor Normal Form of β. Then $\alpha + \beta = \alpha \oplus \beta$.