MA 3205 – Set Theory – Homework for Week 12

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Homework. The homework follows the lecture notes. Below the list of the homeworks for the tutorials from 07.11.2006 onwards. It is not mandatory to hand in homework; but it is recommended to solve the questions by yourself before going to the tutorials. Homework can be corrected on request.

Exercise 16.9. Construct a one-to-one function h which maps $\alpha \times \omega$ to α for any infinite limit ordinal α. This function can without loss of generality assume that the input is of the form $(\gamma \cdot \omega + n, m)$ where $m, n \in \mathbb{N}$ and γ is an ordinal with $S(\gamma) \cdot \omega \leq \alpha$; the image should be of the form $\gamma \cdot \omega + \tilde{h}(n, m)$ for some function $\tilde{h} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.

Exercise 17.7. Let A, B, C be any sets and, as in Example 3.15,

$$D = \{f \in C^A \mid \exists g \in B^A \, \exists h \in C^B \, (f = h \circ g)\}.$$

Show that $D = C^A$ iff $|B| \geq \min\{|A|, |C|\}$.

Exercise 17.12. Consider the following partial ordering given on the set $\mathbb{N}^{\mathbb{N}}$ of all functions from $\mathbb{N}$ to $\mathbb{N}$:

$$f \sqsubset g \Leftrightarrow \exists n \, \forall m > n \, (f(m) < g(m)).$$

This partial ordering only shares some but not all of the properties of the ordering $<_{lin}$ considered above. In order to see this, show the following two properties:

- For countably many functions $f_0, f_1, \ldots$ there is a function g such that $\forall n \in \mathbb{N} \, (f_n \sqsubset g)$;
- There are uncountably many f below the exponential function $n \mapsto 2^n$. Namely for every $A \subseteq \mathbb{N}$ the function $c_A : n \mapsto \sum_{m \in n} 2^{n-m-1} \cdot A(m)$ is below the exponential function.

Note that $c_A \sqsubset c_B \Leftrightarrow A <_{lex} B$. Thus there is an uncountable linearly ordered set of functions below the exponential function.

Exercise 17.14. Use the Axiom of Choice to prove the following: If $|A| = \aleph_1$ and every $B \in A$ satisfies $|B| \leq \aleph_1$ then $|\bigcup A| \leq \aleph_1$.