MA 3205 – Set Theory – On Final Examination

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Format and marks.
The format of the examination will be similar to that of the midterm examinations. There will be 67 marks, so that the sum of the overall marks consists of marks obtained during the continual assessment (up to 33 marks) plus those marks obtained in the final examination. The calculation of the grade based on the marks follows the guidelines of the NUS and takes the distribution of the marks into account.

What to learn for exam.
The questions cover all material in the lecture notes which was presented in the lecture hall.

Review the Midterm Exams.
The material covered there is also part of the final exam, although the questions are of course not the same.

Review Definitions from Lecture Notes.
Review the Definitions and the Axioms of Zermelo and Fraenkel from the lecture notes. Write them down formally, for example:
There is an empty set: $\exists x\, \forall y\, (y \notin x)$.
There are pairs: $\forall x, y\, \exists z\, \forall u\, (u \in z \Leftrightarrow x = u \vee y = u)$.
$(L, \leq)$ is a preordered set iff $\forall x, y, z \in L\, (x \leq y \wedge y \leq z \Rightarrow x \leq z)$.
x is a transitive set iff $\forall y \in x\, \forall z \in y\, (z \in x)$.
An ordinal α is a cardinal iff $\forall \beta \in \alpha\, \forall f : \beta \to \alpha\, (f[\beta] \subset \alpha)$.
Write down the definition of the set of functions from X to Y, the definition of $\mathcal{P}(X)$ and the definition of $X \times Y$.
Review the difference between similar notions, for example between an ordinal and a well-ordered set. Also, find when one object can be expressed in terms of other ones, for example how is the ordered pair defined in terms of unordered pairs? What are the relations between an ordinal α, the set V_α and the rank of members of V_α? When is an ordinal a natural number?

Review basic Facts and Important Theorems from Lecture Notes.
Is the power set of a finite set finite? What about the power set of a countable set? Is there a graph which is neither a partially ordered set nor a function? When is a preordering also a partial ordering? Is Cantor's addition of the addition of ordinals commutative? Which prominent functions defined on the whole class V are constructed by transfinite recursion and how is that done? What do you know about the

Continuum Hypothesis? Are well-ordered sets always isomorphic? What is $\aleph_2 \cdot \aleph_3$? Is there a surjective function from some set X to the universe V?

Train some basic operations with set-theoretic concepts.
Define the ordinal arithmetic by transfinite induction. Put ordinals like $\omega^4 + \omega^5 + \omega^3$, $(\omega + 3) \cdot (\omega + 5)$ into Cantor Normal Form and add and multiply ordinals in Cantor Normal Form.
Determine the cardinality of well-known sets like the following ones: the set of real numbers, the set of the rational numbers, the set of polynomials from natural numbers to themselves, the set of all functions from the natural numbers to themselves, the set of order isomorphisms of the integers, the set of all prime numbers, the set of all natural numbers with up to 512 decimal digits.
Construct a well-ordering on the set of rationals, it is of course not the natural ordering of this set.
Construct functions showing that certain sets have the same cardinality. For example, construct a bijection from $\{F \subseteq \{0, 1, 2, 3, \ldots\} : F \text{ is finite}\}$ to $\{0, 1, 2, 3, \ldots\}$. In some cases, one does also not show directly that $|X| = |Y|$ and would do it more indirectly by showing $|X| \leq |Y|$ and $|Y| \leq |X|$; in these cases, two injective functions from X to Y and from Y to X are constructed.
Construct a dense and countable set which is not isomorphic to the set of rational numbers.
Give an example of a partially ordered set $(P, \sqsubset)$ which is not linearly ordered.
Study the various types of orderings (lexicographic, length-lexicographic, Kleene-Brouwer ordering, ...) on the set of finite sequences $A*$ which are induced by the ordering $<$ on A itself.

Do some easy proofs.
For example, assume that $f : A \to \alpha$ is a one-to-one mapping, α an ordinal and $\forall x, y \in A\,(x \sqsubset y \Leftrightarrow f(x) \in f(y))$. Prove that $(A, \sqsubset)$ is a well-ordered set.
Prove Cantor's result that $\kappa \neq 2^\kappa$.
Prove that there is no bijection from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ of all digits to the set $\{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z\}$ of all letters.

Try to transfer results from the lecture into a similar setting.
According to guidelines, there should be some few questions in the exam which go beyond that what was taught to the students but can be solved with methods similar to those used in the lecture. In order to get mark A, it might be useful to try to modify some theorems and to prove or disprove these results.
For example, let H be the class of all hereditarily countable sets x, that is the class of all x with $\forall y \in \mathcal{TC}(x)\,(|y| \leq \aleph_0)$. Is there an ordinal β such that $H = V_\beta$? Is H identical with the class of those x where $\mathcal{TC}(x)$ is at most countable? Is H actually a set or a proper class?

Additional Exercises.
The following additional exercises can be used to review the recent material.

Exercise 18.3. Show that the standard representation can be defined in set-theory: First define a representation for the set $A = \mathbb{Z} \cup \{sign\}$. Then look at the class of all functions $r : A \to \{0,1,2,3,4,5,6,7,8,9,+,-\}$, call it B; the decimal point could be placed between $r(0)$ and $r(-1)$ and need not to be represented explicitly.

Define which elements of B represent real numbers and get $\mathbb{R}$ by comprehension, state the property explicitly. For this and further definitions, integer constants, integer addition and $<$ on the integers can be used to in order to deal with positions of digits. The selection should be made such that r represents $\sum_{z \in \mathbb{Z}} r(z) \cdot 10^z$ in the case that $r(sign)$ is $+$ and $-\sum_{z \in \mathbb{Z}} r(z) \cdot 10^z$ in the case that $r(sign)$ is $-$. Make sure that every real occurs in the representation exactly once. For example, fix the sign of 0 to either $+$ or $-$.

This representation has the disadvantage that $\mathbb{N} \not\subseteq \mathbb{R}$. So one distinguishes as in many programming languages like FORTRAN between the natural number 2 and the real number 2.0. Nevertheless, there is a one-to-one mapping $f : \mathbb{N} \to \mathbb{R}$ which maps every natural number to its representative in $\mathbb{R}$. f can be defined inductively using a $g : \mathbb{R} \to \mathbb{R}$ such that $f(S(n)) = g(f(n))$ for all natural numbers n. Give two properties $nat, succ$ such that $nat(r)$ is true iff r is in the range of f and $succ(r,q)$ is true if $nat(r) \wedge nat(q) \wedge q = g(r)$.

Exercise 19.4. If $A \subseteq \mathbb{R}$ is at most countable, then $\mathbb{R} - A$ has cardinality $2^{\aleph_0}$.

Exercise 19.7. Cantor's Discontinuum is given as $F(\mathbb{N}^{\{0,2\}})$ where F maps every $f \in \mathbb{N}^{\{0,2\}}$ to the real number having the digits $f(0)f(1)f(2)\ldots$ in the ternary digital representation after the point:

$$F(f) = \sum_{n \in \mathbb{N}} f(n) \cdot 3^{-1-n}.$$

For example, if $f(0) = 2$ and $f(n) = 0$ for all $n \geq 1$ then $F(f)$ represents the ternary number $0.02222\ldots$, that is, $F(f) = \frac{1}{3}$. If E is the set of even numbers and $f(n) = 0$ for $n \in E$ and $f(n) = 2$ for $n \notin E$ then $F(f)$ is the ternary number $0.20202020\ldots$, that is, $F(f) = \sum_{n \in E} 2 \cdot 3^{-1-n} = \frac{3}{4}$. Show that

1. F restricted to $\mathbb{N}^{\{0,2\}}$ is one-to-one;
2. $F(\mathbb{N}^{\{0,2\}})$ does not have any nonempty open subset;
3. $F(\mathbb{N}^{\{0,2\}})$ is perfect.

Furthermore, show that $F(\mathbb{N}^{\{0,2\}})$ is given as

$$\begin{aligned} F(\mathbb{N}^{\{0,2\}}) &= \{r \in \mathbb{R} \mid 0 \leq r \leq 1\} - T \text{ where} \\ T &= \{r \in \mathbb{R} \mid \exists m, n \in \mathbb{N}\, (m \cdot 3^{-n} + 3^{-1-n} < r < m \cdot 3^{-n} + 2 \cdot 3^{-1-n})\}, \end{aligned}$$

that is, T is the set of all positive real numbers for which the digit 1 appears in every ternary representation after the point; $\frac{1}{3} \notin T$ since it has besides $0.1000\ldots$ also the representation $0.02222\ldots$ where no 1 occurs after the point.

Exercise 20.5. Given any model $(V, \in)$, show that $(V_{\omega_1}, \in)$ is not an inner model of ZFC. Take the set $\mathcal{P}(\mathbb{N})$ and show that $\mathcal{P}(\mathbb{N}) \in V_{\omega_1}$. Now let α be its cardinal in $(V_{\omega_1}, \in)$. Then α is also an ordinal in $(V, \in)$ and $|\mathcal{P}(\mathbb{N})| = |\alpha|$ in $(V, \in)$ as well. Show that this contradicts $\alpha < \omega_1$.

Exercise 20.12. Put the following ordinals into Cantor Normal Form:

- $\omega^{\omega+2} \cdot \omega^{\omega+2} \cdot \omega^{\omega+2}$;
- $(\omega + 99) \cdot (\omega^2 + \omega + 1)$;
- $(\omega^{16} + \omega^{12}) \cdot \omega^{\omega}$;
- $(\omega^5 + \omega^4 + \omega^3 + \omega^2 + \omega + 1)^2$;
- $\omega^3 + \omega^2 + \omega^4 + \omega + \omega^5 + 1$;
- $(\omega^3 + \omega) \cdot (\omega^2 + 1)$.