MA 5219 - Logic and Foundations of Mathematics 1
Homework due in Week 5, Tuesday.

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Hand in each starred homework; 1 mark per homework (if it is correct), up to 10 marks in total for homework.

5.1* **Models.** (a) Let X contain the following formulas:

- $\exists x\,[Px]$;
- $\forall x, y, v, w\,[(Px \wedge Py \wedge Pv \wedge Pw \wedge (x \neq v \vee y \neq w)) \rightarrow f(x,y) \neq f(v,w)]$;
- $\forall u, v, w\,[Pv \wedge Pw \rightarrow \neg P(f(v,w))]$;
- $\forall u \exists v, w\,[\neg Pu \rightarrow (Pv \wedge Pw \wedge f(v,w) = u)]$.

Assume that there is a finite set A with a function $f : A \rightarrow A$ and a predicate P on A is given which is a model of X. Let n be the number of elements $a \in A$ satisfying Pa and m be the number of elements $a \in A$ satisfying $\neg Pa$. How do m and n relate to each other?

(b) Make a set Y of formulas using a function f and predicate P such that the m, n from above satisfy $m = 1 + 2 + 3 + \ldots + n$.

5.2 Graphs. A graph G is a base set V with a relation E such that $E(x,y)$ stands for x and y being connected in the graph. A graph (V, E) is called a random graph iff V is infinite and for every two finite disjoint sets C, D of vertices there is a vertex z such that $E(x,z)$ for all $x \in C$ and $\neg E(y,z)$ for all $y \in D$. Make a set X of formulas such that a graph (V, E) satisfies X iff (V, E) is a random graph.

5.3 Matrix Rings. Construct a set X of formulas which enforces that a structure $(R_1, R_2, +, \cdot, 0, 1, det, e_0, e_1, e_2, e_3)$ has the following properties: $(R_1, +, \cdot, 0, 1)$ is a commutative ring with 1 and is a subring of a noncommutative ring $(R_2, +, \cdot, 0, 1)$ with $R_1 \subseteq R_2$ such that $(R_2, +, \cdot, 0, 1)$ is isomorphic to the ring of $2 * 2$-matrices over R_1 and det assigns to every member of R_2 the value which the determinant over the corresponding matrix would have. 0 and 1 are the neutral elements for ring addition and ring multiplication in R_2. The constants e_0, e_1, e_2, e_3 can be used freely to define the structure.

5.4 Dense Linear Orders. Assume that $(A, <, 0, 1)$ is a linearly ordered set satisfying the additional formulas $0 < 1$ and $\forall x, y\,\exists z\,[0 \leq x < z < y \leq 1 \vee 0 \leq y < z < x \leq 1 \vee 0 \leq x = z = y \leq 1]$. Recall that $x \leq y$ abbreviates $x < y \vee x = y$; furthermore, linear orders satisfy the axioms $\forall x, y, z\,[x < y \wedge y < z \rightarrow x < z]$ and $\forall x, y\,[x < y \vee y < x \vee x = y]$ and $\forall x\,[\neg x < x]$. Show that all countable models of this ordering are isomorphic (provided that preassignments to free variables are not taken into account).