MA 5219 - Logic and Foundations of Mathematics 1
Course-Webpage `http://www.comp.nus.edu.sg/~fstephan/mathlogic.html`
Homework due in Week 4, Tuesday 03 September 2013.

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Hand in each homework which you want to be checked; 1 mark per each correct starred homework; up to 10 marks in total for homework - there will be more than 10 starred homeworks, so you have several chances to try.

4.1 Proof systems. Consider the following rules.

$$\frac{\emptyset}{A \vdash A} \qquad \frac{X \vdash A}{X \cup Y \vdash A}$$
$$\frac{X \vdash A, A \to B}{X \vdash B} \qquad \frac{X, A \vdash B}{X \vdash A \to B}$$
$$\frac{X \vdash A \to \bot}{X \vdash \neg A} \qquad \frac{X \vdash \neg A}{X \vdash A \to \bot}$$
$$\frac{X \vdash \neg(A \to \neg B)}{X \vdash A, B} \qquad \frac{X \vdash A, B}{X \vdash \neg(A \to \neg B)}$$
$$\frac{X \vdash \neg\neg A}{X \vdash A} \qquad \frac{X \vdash A}{X \vdash \neg\neg A}$$
$$\frac{X \vdash \neg A \to B}{X \vdash \neg B \to A} \qquad \frac{X, A \vdash B \text{ and } X, \neg A \vdash B}{X \vdash B}$$

Derive the following rules from the above rules.

$$\frac{X \vdash A \to B, \neg A \to B}{X \vdash B} \quad \frac{X \vdash A \to B \to C}{X \vdash B \to A \to C} \quad \frac{\emptyset}{X \vdash A \to B \to A}$$

4.2* Operators. Is there a structure $(A, \circ)$ satisfying the law of commutativity and the below law of inversion though $\circ$ does not need to have a neutral element, that is, $(A, \circ)$ satisfies only the first of the following two laws:

$$\forall a, b \in A \; \exists c \in A \quad [a \circ b = b \circ a \wedge a \circ c = b];$$
$$\exists e \in A \; \forall a \in A \quad [e \circ a = a \circ e = a]?$$

Note that $\circ$ is not required to be associative. Prove your answer.

4.3* Ordered Semigroups. Assume that $(A, \circ, \leq)$ is an ordered semigroup such that $\circ$ is associative, $\leq$ is transitive and the following two rules hold:

$$\forall a, b \in A \, [a \leq b \wedge b \leq a \Leftrightarrow a = b]; \; \forall a, b, c \in A \, [a \leq b \Rightarrow a \circ c \leq b \circ c \wedge c \circ a \leq c \circ b].$$

Does every model of such a semigroup have an element b with $\forall a \in A \, [a \circ b = b \circ a]$?