MA 5219 - Logic and Foundations of Mathematics 1
Course-Webpage `http://www.comp.nus.edu.sg/~fstephan/mathlogic.html`
Homework due in Week 13, Tuesday 12 November 2013.

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Hand in each homework which you want to be checked; 1 mark per each correct starred homework; up to 10 marks in total for homework - there will be more than 10 starred homeworks, so you have several chances to try.

13.1* Axioms of Integers.
Provide a finite set X of axioms for addition $+$, subtraction $-$ and an ordering $<$ plus a set of terms of the form $0,\ 1,\ 1+1,\ 1+1+1,\ \ldots$ and $-1,\ -1-1,\ -1-1-1,\ \ldots$ such that for any model of X it holds that a member a of the domain A of the model is either equal to a term $\underline{z}$ of above form ($z \in \mathbb{Z}$) or satisfies $a < \underline{z}$ for all $z \in \mathbb{Z}$ or satisfies $a > \underline{z}$ for all $z \in \mathbb{Z}$.

13.2* Axioms Q1–Q5 of Successor and Addition.
Recall the axioms Q1–Q5 from page 234 (without multiplication):
Q1: $\forall x\, [Succ(x) \neq 0]$;
Q2: $\forall x \forall y\, [Succ(x) = Succ(y) \rightarrow x = y]$;
Q3: $\forall x \exists y\, [x = 0 \vee x = Succ(y)]$;
Q4: $\forall x\, [x + 0 = x]$;
Q5: $\forall x \forall y\, [x + Succ(y) = Succ(x + y)]$.
Is there a model of Q1–Q5 such that addition is not commutative in this model, that is, is there a model with elements i, j satisfying $i + j \neq j + i$.

13.3* Primitive Recursive Functions. Recall that a function $f : \mathbb{N} \rightarrow \mathbb{N}$ is primitive recursive iff there is a computer program using if-then-else statements, for-statements (where the loop body does not modify any variables mentioned in the header of the loop-statement), functions (with the command "Return(..)" to return from a function with a value), addition, multiplication, division (downrounded), remainder, subtraction (with $2 - 5 = 0$ to avoid negative numbers) and order; furthermore none of the functions used should call itself directly or indirectly. Show that the following function are primitive recursive by giving the corresponding programs: (a) $n \mapsto 2^n$, (b) $n \mapsto \min\{m > n : m \text{ is prime}\}$, (c) $n \mapsto p_n$ where $p_0 = 2, p_1 = 3, p_2 = 5, p_3 = 7, \ldots$ and, in general, p_n is the n-th prime in ascending order, (d) $n, m \mapsto k$ where $n > 0$ and k is the largest number such that $(p_m)^k$ divides n. For (b) and (c) note that for each $n > 0$ there is a pirme number between n and $2n$; this can be used to bound the for-loop.

13.4 Recursively enumerable sets. Show that for an infinite set $A \subseteq \mathbb{N}$, the following conditions (describing r.e. sets) are equivalent: A is the domain of a partial-recursive function; A is the range of a recursive function; A is the range of a primitive-recursive function; A is the range of a recursive one-one function.