MA 4207 - Mathematical Logic
Course-Webpage `http://www.comp.nus.edu.sg/~fstephan/mathlogicug.html`
Homework due in Week 7, Monday 2 March 2015

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Homework 7.1
Assume that there are infinitely many logical atoms. Is there a set S of formulas such that for all v mapping atoms to $\{0,1\}$, $v \models S$ iff there are exactly three atoms A, B, C with $v(A) = 1, v(B) = 1, v(C) = 1$?

Homework 7.2
For first-order logic, assume that the logical language has only equality and variables and quantifiers and the logical connectives. The formula $\exists x, y, z\, [x \neq y \wedge x \neq z \wedge y \neq z]$ can only be satisfied by a structure with at least three elements. Is there, in this logical language, a formula α which can only be satisfied by structures with infinitely many elements? Is there a set of such formulas?

Homework 7.3
Consider the structure $(\mathbb{N}, +, -, \cdot, <, =, 0, 1, 2, \ldots)$ and the corresponding first-order logical language of arithmetic with constants for every natural number. Make formulas which express the following:

1. Each number is either 0 or 1 or the multiple of a prime number;
2. There are infinitely many prime numbers;
3. Every even number other than 0 and 2 is the sum of two prime numbers;
4. The number 23 is not the sum of three squares;
5. Every number is the sum of four squares;
6. There are infinitely many numbers x such that $x - 1$ and $x + 1$ are both prime numbers.

Homework 7.4
Let $(F, +, -, \cdot, f, =, 0, 1, 2)$ be the finite field with the three elements $0, 1, 2$ and a one-place function f. Which of the following statements are true for this structure?

1. $\forall x, y\, [(x + y) \cdot (x + y) = (x \cdot x) + (y \cdot y) - (x \cdot y)]$;
2. $\forall x, y\, [(x + y) \cdot (x + y) \cdot (x + y) = (x \cdot x \cdot x) + (y \cdot y \cdot y)]$;
3. $\forall x, y\, [(x + y) \cdot (x + y) \cdot (x + y) \cdot (x + y) = (x \cdot x \cdot x \cdot x) + (y \cdot y \cdot y \cdot y)]$;
4. $\exists a, b, c\, \forall x\, [f(x) = a \cdot x \cdot (x - 1) + b \cdot x \cdot (x - 2) + c \cdot (x - 1) \cdot (x - 2)]$;
5. $\forall x\, [x \cdot x \cdot x \neq 2]$.