MA 4207 - Mathematical Logic
Homework to be done in Week 8 (this side) and due in Week 9 (back side), Monday
16 March 2015
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Convention: The homeworks 8.1, 8.2, 8.3 and 8.4 use the fixed finite structure of
calculations modulo 10 with addition and multiplication; so 5 + 8 = 3 and 7 * 7 = 9.
Please read the e-learning handout before doing the homework; the homeworks will
be done interactively in the tutorial next week through WebEx.

Homework 8.1
Either write a parser and translate the following formulas over the arithmetic modulo
10 into computer programs of write directly the corresponding programs. You might
use Firefox Scratchpad to test the programs. Free variables are parameters of the
program, when calling them you can either set a default-value or ask the user to enter
the value.

- \( \exists x [3 = x + x + 1]; \)
- \( \exists x \exists y [3 = (3 * x + 5 * y) * (5 * x + 3 * y)]; \)
- \( \exists y [x = y + y + 1]; \)
- \( \forall x \exists y [y * z = x]; \)
- \( \forall x \exists y \forall z [z * z \neq (x + y) * 3]. \)

Homework 8.2
Formalise the following in a formula and then translate it into a program: There is a
polynomial \( f \) of degree 3 or less in \( x \) such that \( f(x) = x^{31} + x^{33} \) for all choices of \( x \) in \( A \).

Homework 8.3
Try to generalise the result of 8.2 by showing that there is an \( m > 0 \) with \( x^{n+m} = x^n \)
for all \( n > 0 \). Once \( m \) is found, it is enough to verify the above for \( n = 1, 2, \ldots, m \).

Homework 8.4
Check with a formula whether there is any polynomial which maps all non-zero num-
bers to 1 and 0 to 0. Translate it into a program and run the program to check
whether this formula exists. Use the result of Exercise 8.3 when making the formula
and program.
These homeworks are due in week 9.

**Homework 9.1**
Let \((A, +, \ast)\) and \((B, +, \ast)\) be the remainder rings modulo \(a\) and \(b\), respectively, \(a, b \in \{2, 3, \ldots\}\). For which \(a, b\) is there a homomorphism \(f\) from \((A, +, \ast)\) to \((B, +, \ast)\) such that any two terms \(t_1, t_2\) satisfy \((A, +, \ast) \models t_1 = t_2\) iff \((B, +, \ast) \models f(t_1) = f(t_2)\).

**Homework 9.2**
Choose values for \(a, b\) from Homework 9.1 and a formula \(\phi\) such that \(f\) is a homomorphism and the formula \(\phi\) is true in \((A, +, \ast)\) but not in \((B, +, \ast)\).

**Homework 9.3**
Let \(\mathbb{Z} \ast \{i\} + \mathbb{Z}\) be the set of all complex integer numbers. Show that this set together with \(+\) and \(\ast\) is a ring. Prove that the basis element \(i\) is not definable by using a homomorphism which maps \(i\) to some other element.

**Homework 9.4**
Assume that \((A, +, \ast, 0, 1)\) is a finite ring with \(0 \neq 1\). Consider the formulas
\[
\begin{align*}
x = 0 & \iff \forall y \left[x + y = y\right] \text{ and } \forall y \left[x \ast y = y \land y \ast x = y\right].
\end{align*}
\]
Are then all members of \(A\) definable with formulas like this? If yes then prove how this is done else provide a finite ring where some elements are not definable.

**Homework 9.5**
Let \((\mathbb{R}, +, \ast, <, 0, 1)\) be the ordered field of the real numbers with the constants 0 and 1. Prove that all rational numbers and all real roots of polynomials are defineable. Provide then examples of formulas \(\phi_1, \phi_2, \phi_3, \phi_4\) such that \(x_k\) is the unique element satisfying \(\phi_k\) where the formulas \(\phi_k\) say the following:
\[
\begin{enumerate}
\item \(x_1 = 2/3;\)
\item \(x_2\) is the positive square-root of 3;
\item \(x_3\) is the largest number satisfying \(x_3^{10} - 4x_3^5 + 2 = 0;\)
\item \(x_4\) is the smallest number satisfying \(3x_4^6 - 6x_4^4 + 3x_4^2 = 0.\)
\end{enumerate}
\]