MA 4207 - Mathematical Logic
Course-Webpage `http://www.comp.nus.edu.sg/~fstephan/mathlogicug.html`
Homework for training purposes only.

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Homework 14.1
Give an example for a set S of formulas in sentential logic such that

- for all $\alpha, \beta \in S$, the formulas $(\alpha \vee \beta), (\alpha \wedge \beta)$, $(\alpha \rightarrow \beta)$, $(\alpha \leftrightarrow \beta)$ and $\neg\neg\alpha$ are also in S;
- for all α, either $\alpha \in S$ or $\neg\alpha$ in S but not both.

Solution. Let v be such that $v(A) = 1$ for all atoms A. Now let

$$S = \{\alpha : \overline{v}(\alpha) = 1\}$$

and one can see that due to the definition of $\overline{v}$, it is always true that exactly one of $\alpha, \neg\alpha$ are in S. Furthermore, if α, β are in S then $\overline{v}$ makes both of them true and it follows that $\overline{v}$ also makes the formulas $(\alpha \vee \beta), (\alpha \wedge \beta)$, $(\alpha \rightarrow \beta)$, $(\alpha \leftrightarrow \beta)$ and $\neg\neg\alpha$ true, thus they are also in S.

Homework 14.2
Prove by induction that for every formula using only $\oplus$, $\leftrightarrow$ and $\neg$ as connectives, which is built from the atoms $A_1, A_2, \ldots, A_n$, either all possible assignments of these n values or half of them or none of them evaluates the formula to true.

Solution. What one is proving by induction is the following statement: Given a formula α using the above indicated connectives, one defines $\mathit{Depend}(\alpha)$ to be the set of all atoms A such that there are v, w assigning values to the atoms different only on A_k with $\overline{v}(A_k) \neq \overline{w}(A_k)$. One shows now by induction the following statement for formulas α of the given type:

$(*)$ If v, w differ exactly on A_k and $A_k \in \mathit{Depend}(\alpha)$ then $\overline{v}(\alpha) = \neg\overline{w}(\alpha)$.

To see $(*)$, first note that for constants, the sets $\mathit{Depends}(0)$ and $\mathit{Depends}(1)$ are empty and thus the statement is true; furthermore, if $\alpha = A_k$ then $\mathit{Depend}(A_k) = A_k$ and it is obvious that if v, w differ on A_k then $\overline{v}(A_k) = \neg\overline{w}(A_k)$. Now for an induction, consider α, β which are satisfying $(*)$:

1. Consider $\gamma = \neg\alpha$. One let $\mathit{Depend}(\gamma) = \mathit{Depend}(\alpha)$ and considers any v, w which differ only in one atom A_k: If $A_k \in \mathit{Depend}(\alpha)$ then $\overline{v}(\gamma) = \neg\overline{v}(\alpha) = \neg\neg\overline{w}(\alpha) = \neg\overline{w}(\gamma)$; If $A_k \notin \mathit{Depend}(\alpha)$ then $\overline{v}(\gamma) = \neg\overline{v}(\alpha) = \neg\overline{w}(\alpha) = \overline{w}(\gamma)$. So for v, w only differing in A_k, $\overline{v}(\gamma) = \neg\overline{w}(\gamma)$ iff $A_k \in \mathit{Depend}(\gamma)$.

2. Consider $\gamma = \alpha \oplus \alpha$. One let *Depend*$(\gamma)$ be the symmetric difference of *Depend*(α) and *Depend*(β), that is, contain exactly those atoms which are in exactly one of the sets *Depend*(α) and *Depend*(β). Consider any v, w which differ only in one atom A_k: If $A_k \in Depend(\gamma)$ then A_k is exactly in one of the sets *Depend*(α) and *Depend*(β), without loss of generality say in the first. Now $\overline{v}(\alpha)$ and $\overline{w}(\alpha)$ differ while $\overline{v}(\beta)$ and $\overline{w}(\beta)$ are the same. It follows that one of $\overline{v}(\alpha \oplus \beta)$, $\overline{w}(\alpha \oplus \beta)$ is $0 \oplus c$ while the other one is $1 \oplus c$, where $c \in \{0, 1\}$. Hence $\overline{v}(\gamma) = \neg\overline{w}(\gamma)$. If $A_k \notin Depend(\gamma)$ by $A_k \notin Depend(\alpha), A_k \notin Depend(\beta)$ then $\overline{v}(\alpha) = \overline{w}(\alpha)$, $\overline{v}(\beta) = \overline{w}(\beta)$ and $\overline{v}(\gamma) = \overline{v}(\alpha \oplus \beta) = \overline{w}(\alpha \oplus \beta) = \overline{w}(\gamma)$. If $A_k \notin Depend(\gamma)$ by $A_k \in Depend(\alpha), A_k \in Depend(\beta)$ then $\overline{v}(\alpha) = \neg\overline{w}(\alpha)$, $\overline{v}(\beta) = \neg\overline{w}(\beta)$ and $\overline{v}(\gamma) = \overline{v}(\alpha \oplus \beta) = \overline{w}(\neg\alpha \oplus \neg\beta) = \overline{w}(\alpha \oplus \beta) = \overline{w}(\gamma)$. So for v, w only differing in A_k, $\overline{v}(\gamma) = \neg\overline{w}(\gamma)$ iff $A_k \in Depend(\gamma)$.

3. If $\gamma = \alpha \leftrightarrow \beta$ then again *Depend*(γ) is the symmetric difference of *Depend*(α) and *Depend*(β). The proof in this case is the same as in the case of $\oplus$; alternatively, one could also replace $\alpha \leftrightarrow \beta$ by $\neg(\alpha \oplus \beta)$ and do the two prior inductive steps.

Thus the induction gives that for each formula α there are two cases: Either *Depend*(α) $= \emptyset$ and then the truth-table of α assigns in all rows the same value or *Depend*(α) contains at least one atom A_k and if one puts this atom A_k into the last column of the truth-table and on can group the rows in pairs of rows where the truth-entries differ only for A_k and thus one of these rows carries the value 0 while the other one carries the value 1; hence half of the rows has a 0 and half has a 1. Here an example for $A_h \oplus \neg(A_k \oplus A_h)$:

A_h	A_k	$A_h \oplus \neg(A_k \oplus A_h)$
0	0	1
0	1	0
1	0	1
1	1	0

Homework 14.3
Make a formal proof that

$$\forall x \, \forall y \, [\alpha \to \beta] \to \forall y \, \forall x \, [\neg\beta \to \neg\alpha]$$

is a valid formula.

Solution. Recall that tautologies in sentential logic can be made to axioms in first-order logic by replacing the atoms by logical symbols; furthermore, any formula of the form $\forall x \, [\gamma] \to \gamma$ is an axiom. Thus one can make the following proof.

1. $\{\forall x \, \forall y \, [\alpha \to \beta]\} \vdash \forall x \, \forall y \, [\alpha \to \beta]$ (Copy)
2. $\{\forall x \, \forall y \, [\alpha \to \beta]\} \vdash \forall x \, \forall y \, [\alpha \to \beta] \to \forall y \, [\alpha \to \beta]$ (Axiom)
3. $\{\forall x \, \forall y \, [\alpha \to \beta]\} \vdash \forall y \, [\alpha \to \beta]$ (Modus Ponens)
4. $\{\forall x \, \forall y \, [\alpha \to \beta]\} \vdash \forall y \, [\alpha \to \beta] \to \alpha \to \beta$ (Axiom)
5. $\{\forall x \, \forall y \, [\alpha \to \beta]\} \vdash \alpha \to \beta$ (Modus Ponens)

6. $\{\forall x\, \forall y\, [\alpha \rightarrow \beta]\} \vdash (\alpha \rightarrow \beta) \rightarrow (\neg\beta \rightarrow \neg\alpha)$ (Axiom)
7. $\{\forall x\, \forall y\, [\alpha \rightarrow \beta]\} \vdash \neg\beta \rightarrow \neg\alpha$ (Modus Ponens)
8. $\{\forall x\, \forall y\, [\alpha \rightarrow \beta]\} \vdash \forall x\, [\neg\beta \rightarrow \neg\alpha]$ (Generalisation Theorem)
9. $\{\forall x\, \forall y\, [\alpha \rightarrow \beta]\} \vdash \forall y\, \forall x\, [\neg\beta \rightarrow \neg\alpha]$ (Generalisation Theorem)
10. $\emptyset \vdash \forall x\, \forall y\, [\alpha \rightarrow \beta] \rightarrow \forall y\, \forall x\, [\neg\beta \rightarrow \neg\alpha]$ (Deduction Theorem)

Homework 14.4
Is the statement

$$\{Px \rightarrow Py\} \vdash \forall z\, [Px \rightarrow Pz]$$

correct? If so, make a formal proof, if not, make a model with default values of the variables for which it is false.

Solution. This statement is not correct. Assume that a model is given with variable defaults, that the model has at least the values $0, 1$, that $P(x)$ is equivalent to $x = 0$ and that x, y have the default value 0. Then $Px \rightarrow Py$ is true and $Px \rightarrow P1$ is false; in particular $\forall z\, [Px \rightarrow Pz]$ is false.

Homework 14.5
Choose a logical language and a theory T in this language such that

- T is finite axiomatisable;
- T is $\aleph_0$-categorical and $\aleph_1$-categorical;
- T has a finite model of m elements iff $m = 3^n$ for some n.

Furthermore, is T complete? Explain your answer.

Solution. The idea is to use the language of Abelian groups where an element three times added to itself gives 0. These structures are equivalent to vector spaces over $\mathbb{F}_3$ and it is known from linear algebra that each two such vector spaces are isomorphic iff they are vector spaces over the same field and their bases have the same cardinality. Note that scalar multiplication over $\mathbb{F}_3$ with 0 gives the constant 0 function and with 1 gives the identity function and with 2 gives the sum of an element with itself. Thus one can define scalar multiplication by cases and does not need to incorporate it into the logical language. So the only symbols added into the language are 0 (neutral element) and + (addition modulo 3 in a vector space). The axioms postulated are now the following ones:

1. $\forall x\, \forall y\, \forall z\, [(x + y) + z = x + (y + z)]$;
2. $\forall x\, \forall y\, [x + y = y + x]$;
3. $\forall x\, [x + 0 = x]$;
4. $\forall x\, [x + (x + x) = 0]$.

Now, if κ is an infinite cardinal then, by results of linear algebra, a vector space over $\mathbb{F}_3$ has a basis of cardinality κ iff the vector space itself has cardinality κ; thus every such vector space and, therefore, also every structure satisfying the above axioms is κ-categorical; in particular these structures are $\aleph_0$-categorical and $\aleph_1$-categorical. Furthermore, the finite vector spaces of dimension n have all 3^n elements and every finite vector space has a finite basis (dimension). It is not needed for this homework to reprove the facts known from basic lectures like linear algebra.