MA 4207 - Mathematical Logic
Course-Webpage `http://www.comp.nus.edu.sg/~fstephan/mathlogicug.html`
Homework due in Week 3.

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Homework 3.1
Cantor's function $x, y \mapsto (x+y)\cdot(x+y+1)/2+y$ is a bijection from $\mathbb{N}\times\mathbb{N}$ onto $\mathbb{N}$. Construct a bijection from $\mathbb{Z}\times\mathbb{Z}$ onto $\mathbb{Z}$.

Homework 3.2
Prove that there is no set X such that its powerset $\{Y : Y \subseteq X\}$ has 5 elements.

Homework 3.3
Show that a power set has always more elements than the given set, that is, fill out the missing details at the following proof-sketch. Recall that $|A| \leq |B|$ iff there is a one-one function from A to B and show that $|\mathcal{P}A| \not\leq |A|$.

Proof-Sketch: The $\emptyset$ has 0 and $\mathcal{P}\emptyset$ has one element, namely $\emptyset$, hence one cannot have a one-one mapping from $\mathcal{P}\emptyset$ to $\emptyset$. Now assume that A is not empty and $f : A \rightarrow \mathcal{P}A$ is a function. Show that there is a set $B \subseteq A$ which is not in the range of f. Then consider any function $g : \mathcal{P}A \rightarrow A$ and prove that this function cannot be one-one, as otherwise a surjective f from A to $\mathcal{P}A$ would exist. Hence $|\mathcal{P}A| \not\leq |A|$.

Homework 3.4
Use Homework 3.3 to prove that there is no set X such that its powerset has as many elements as $\mathbb{N}$. The fact that every set X is either finite or satisfies $|\mathbb{N}| \leq |X|$ can be used in the proof.

Homework 3.5
Let $f(n)$ be the maximum number of negation symbols in a well-formed formula which does not contain any subformula of the form $(\neg(\neg\alpha))$ and which contains at most n atoms. Here $(\neg(A_1 \vee (\neg(A_2 \vee (\neg A_1)))))$ has 3 atoms and n is 3, as repeated atoms are counted again. Determine the value $f(n)$ in dependence of n.

Homework 3.6
Prove by induction that a well-formed formula of length n contains less than $n/3$ connectives and at most $(n+3)/4$ atoms.

Homework 3.7
Use the truth-table method to prove that the following formulas are equivalent:

- $((\neg A_1) \vee (\neg A_2))$;
- $(\neg(A_1 \wedge A_2))$;
- $((A_1 \vee A_2) \leftrightarrow (A_1 \oplus A_2))$.

Homework 3.8
List out the truth-table for the formula $((A_1 \oplus A_2) \wedge (\neg A_3))$.

Homework 3.9
Consider the following formulas:

$$\begin{aligned}
\phi_1 &= (((A_1 \vee A_2) \vee A_3) \wedge ((A_4 \vee A_5) \vee A_6)); \\
\phi_2 &= (((A_1 \vee A_2) \wedge (A_3 \vee A_4)) \wedge (A_5 \vee A_6)); \\
\phi_3 &= (((((A_1 \oplus A_2) \oplus A_3) \oplus A_4) \oplus A_5) \oplus A_6).
\end{aligned}$$

There are $2^6 = 64$ ways to assign the truth-values to the sentence symbols (or atoms) $A_1, \ldots, A_6$. Determine for each of the formulas ϕ_1, ϕ_2, ϕ_3, how many of these assignments make the formula true and how many of these assignments make the formula false.

Homework 3.10
For the formulas from Homework 3.9, is the statement

$$\{\phi_1, \phi_2, \phi_3\} \models (((((A_1 \wedge A_2) \wedge A_3) \wedge A_4) \wedge A_5) \wedge A_6)$$

true or false? Prove your answer.

Homework 3.11
For the formulas from Homework 3.9, is the statement

$$\{\phi_1, \phi_2, \phi_3\} \models (((((A_1 \vee A_2) \vee A_3) \vee A_4) \vee A_5) \vee A_6)$$

true or false? Prove your answer.

Homework 3.12
Using the connectives $\vee, \wedge, \rightarrow, \leftrightarrow, \oplus, \neg$, construct a formula using atoms A_1, A_2, A_3, A_4 which says that at least two and at most three of these atoms are true.

Homework 3.13
Using the connectives $\vee, \wedge, \rightarrow$, construct a formula using atoms $A_1, A_2, A_3, A_4, A_5, A_6$ which says that at all six atoms are either false or all six atoms are true.