MA 4207 - Mathematical Logic
Course-Webpage `http://www.comp.nus.edu.sg/~fstephan/mathlogicug.html`
Homework due in Week 4.

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Homework 4.1
Let $atom(\phi)$ be the set of atoms used in ϕ, so $atom(((A_1 \vee A_2) \wedge A_2)) = \{A_1, A_2\}$ and $atom((0 \vee 1)) = \emptyset$. Let WFF be the set of well-formed formulas. Let

$$C_1 = \{\phi \in WFF : \forall v\, [\text{if } v(A) = 1 \text{ for some } A \in atom(\phi) \text{ then } \overline{v}(\phi) = 1]\}.$$

For which of the connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \oplus$ is C_1 closed under the connective? Here one says that C is closed under the connective $\oplus$ if all formulas $\phi, \psi \in C$ satisfy that $(\phi \oplus \psi) \in C$. Similarly for other connectives.

Homework 4.2
Let $atom(\phi)$ and WFF be defined as in Homework 4.1. Let

$$C_2 = \{\phi \in WFF : \forall v\, [\text{if } v(A) = 0 \text{ for at most one } A \in atom(\phi) \text{ then } \overline{v}(\phi) = 1]\}.$$

For which of the connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \oplus$ is C_2 closed under the connective?

Homework 4.3
Let $C_3 = \{\phi \in WFF$: every $A \in atom(\phi)$ occurs in ϕ exactly once and $0, 1$ do not occur in $\phi\}$. Prove by induction that C_3 does not contain any tautology and also not contain any antitautology. Here a tautology is a formula which is always true (independent of the choice of the truth-values of the atoms) and an antitautology is a formula which is always false.

Homework 4.4
Define on WFF by recursion the functions $atom$ ($\phi \mapsto atom(\phi)$) and $maxatom$ ($\phi \mapsto \max\{k : A_k \in atom(\phi)\}$), where the atoms $A_1, A_2, \ldots$ can be used and $\max \emptyset = 0$. So $maxatom((A_1 \vee (A_4 \wedge A_5))) = 5$ and $maxatom((0 \vee 1)) = 0$.

Homework 4.5
Define on WFF by recursion the function $numcon(\phi)$ as the number of connectives $\wedge, \vee, \rightarrow, \leftrightarrow, \oplus$ occurring in ϕ. Furthermore define the function $numneg(\phi)$ as the number of negations occurring in ϕ. Determine the best-possible constants c, m, n such that

$$|\phi| \leq c \cdot numcon(\phi) + n \cdot numneg(\phi) + m$$

for all WFF ϕ.

Homework 4.6
Let $C_6 = \{\phi \in WFF$: every $A \in atom(\phi)$ occurs in ϕ exactly once$\}$; note that formulas in C_6 might have occurrences of the constants 0 and 1. Define by recursion a function F from C_6 into the rational numbers between 0 and 1 which returns for

each formula $\phi \in C_6$ the truth-probability $n/2^m$ where n is the number of rows in the truth-table of ϕ evaluated to 1 and m is the number of atoms used in the formula so that 2^m is the overall number of rows in the truth-table of ϕ. For example, $F(1) = 1$, $F((A_1 \oplus (A_2 \vee A_3))) = 1/2$ and $F(((A_2 \vee A_5) \wedge (A_3 \vee 0))) = 3/8$.

Homework 4.7
Let $C_7 = \{\phi \in WFF : \phi$ can use the constants $0, 1$ and the only connectives in ϕ are $\wedge$ and $\vee\}$. Prove by induction that a formula $\phi \in C_7$ is a tautology iff $\overline{v}(\phi) = 1$ for the truth-assignment v with $v(A_k) = 0$ for all k.

Homework 4.8
Let $C_8 = \{\phi \in WFF : \phi$ can use the constants $0, 1$ and the only connectives in ϕ are $\wedge$ and $\vee\}$. Prove by induction that a formula $\phi \in C_8$ is an antitautology iff $\overline{v}(\phi) = 0$ for the truth-assignment v with $v(A_k) = 1$ for all k.

Homework 4.9
Let U be a finite set of atoms and $C_9 = \{\phi \in WFF : atom(\phi) \subseteq U\}$. Prove that there is a finite set F of formulas such that for every formula $\phi \in C_9$ there is a $\psi \in F$ with $(\psi \leftrightarrow \phi)$ being a tautology.

Homework 4.10
The following formulas have brackets omitted according to the rule that the binding strengths of the connectives is ordered as $\neg, \wedge, \vee, \oplus, \rightarrow, \leftrightarrow$. Insert back the needed brackets for getting a member of WFF.

1. $A_1 \wedge \neg A_2 \vee A_3 \rightarrow A_4 \wedge \neg A_5$;

2. $A_1 \vee \neg A_2 \wedge \neg A_3 \leftrightarrow A_4 \rightarrow A_5$;

3. $\neg\neg A_1 \vee \neg A_2$.

Homework 4.11
Is there a formula using the connectives $\oplus$ and $\neg$ but no other connectives where the value of the formula depends on the placement of brackets?

Homework 4.12
Let $v(A_1) = 1$, $v(A_2) = 1$, $v(A_3) = 0$. The below formulas are given in Polish notation. Write them as WFF and evaluate them according to v:

1. $\neg \leftrightarrow \oplus A_1 A_2 A_3$;

2. $\wedge \vee \neg \vee A_1 A_2 A_3 A_1$;

3. $\oplus \wedge A_1 A_2 \wedge A_2 A_3$.

Homework 4.13
Write the following formula in Polish notation: $\neg((A_1 \vee \neg A_2) \wedge (A_2 \vee \neg A_3) \wedge (A_3 \vee \neg A_1))$.