MA 4207 - Mathematical Logic
Course-Webpage `http://www.comp.nus.edu.sg/~fstephan/mathlogicug.html`
Homework due in Week 5; can be presented in Week 6 as well.

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Homework 5.1
Assume that a company uses only chips which output 0 when all inputs are 0 – as all-0-inputs and outputs are considered as an "error-information" and every useful information is coded by input-vectors which are not everywhere 0. Now a vendor offers to produce the chips as specified using only "exclusive-or-gates" ($\oplus$) and "inclusive-or-gates" ($\vee$) at a very competitive price. The company boss finds it suspicious and asks the company's technician: Can this work? Provide the correct answer and prove why it can work or why it cannot work.

Homework 5.2
Which of the following statements are true? Prove your answers.
(a) $\{\alpha, \beta\} \models c \vee d \Leftrightarrow \{\alpha, \beta\} \models c$ or $\{\alpha, \beta\} \models d$.
(b) $\{\alpha, \beta\} \models c \wedge d \Leftrightarrow \{\alpha, \beta\} \models c$ and $\{\alpha, \beta\} \models d$.
(c) $\{\alpha, \beta\} \models \alpha \oplus \beta \Leftrightarrow \alpha \wedge \beta$ is not satisfiable.
Here a formula α is satisfiable iff there is a choice of truth-values of the atoms such that α becomes true.

Homework 5.3
Which of the following statements are true? Prove your answers.
(a) $S \models \alpha \Leftrightarrow S \cup \{\alpha\}$ is satisfiable.
(b) $S \models \alpha \Leftrightarrow S \cup \{\neg\alpha\}$ is not satisfiable.
(c) $S \models \alpha \rightarrow \beta \Leftrightarrow S \cup \{\neg\alpha\} \models \neg\beta$.
Here a set S of formulas is satisfiable iff there is a choice of truth-values of the atoms such that all formulas in S are true.

Homework 5.4
Make an infinite set S of formulas such that every subset of two formulas is satisfiable but no subsetset of three or more formulas is.

Homework 5.5
Is the set $\{\leftrightarrow, \neg, \oplus, 0, 1\}$ of connectives and constants complete? Do the subsets $\{\leftrightarrow, 1\}$ and $\{\leftrightarrow, 0\}$ have the same expressive power or less expressive power than $\{\leftrightarrow, \neg, \oplus, 0, 1\}$?

Homework 5.6
Let C_6 consist of all formulas which are atoms or which are formed from other formulas $\alpha, \beta, \gamma \in C_6$ by taking $maj(\alpha, \beta, \gamma)$ or $\neg\alpha$. Prove by induction that each Boolean function B^n_α formed from an $\alpha \in C_6$ satisfies $B^n_\alpha(x_1, \ldots, x_n) = \neg B^n_\alpha(\neg x_1, \ldots, \neg x_n)$ for $x_1, \ldots, x_n \in \{0, 1\}$. Is there expressive power gained by adding $\neg$ into the connectives permitted in C_6?

Homework 5.7
How many Boolean functions can be formed by using input variables $x_1, \ldots, x_n$ and the constants and connectives from $\{0, 1, \wedge\}$.

Homework 5.8
How many Boolean functions can be formed using input variables $x_1, \ldots, x_n$ and the constants and connectives from $\{0, 1, \neg, \oplus\}$.

Homework 5.9
Use as few of "and" ($\wedge$) and "inclusive or" ($\vee$) as possible in order to make a formula α with four atoms A_1, A_2, A_3, A_4 such that the following conditions hold:

- If at least three of the atoms A_1, A_2, A_3, A_4 are true then α is true;
- If at most one of the atoms A_1, A_2, A_3, A_4 are true then α is false;
- If exactly two of the atoms A_1, A_2, A_3, A_4 are true then there is no constraint on which value α takes.

Use the last condition in order to optimise the number of connectives in the formula.

Homework 5.10
Let F be the set of all Boolean formulas with input variables x_1, x_2, x_3, x_4 which are 0 when at most one input variable is 1 and which are 1 when at least three input variables are 1. So F contains functions B^4_α for formulas α like $(A_1 \vee A_2) \wedge (A_3 \vee A_4)$. Which of the following sets of connectives satisfy to generate all formulas in F (plus some outside F): $\{\wedge, \vee\}$, $\{\wedge, \oplus\}$, $\{\oplus, \leftrightarrow\}$?

Homework 5.11
Is there an $n \in \{1, 2, 3, 4\}$ for which the set $\{maj, B^n_{A_1 \oplus A_2 \oplus \ldots \oplus A_n}\}$ complete? For those n where it is incomplete, can it be made complete by adding the logical constants $0, 1$ to the set of connectives? If so, which of these are needed?