MA 4207 - Mathematical Logic
Course-Webpage `http://www.comp.nus.edu.sg/~fstephan/mathlogicug.html`
Homework due in Week 11 (for both tutorial groups).

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Homework 11.1
Assume that α, β, γ are well-formed formulas. Give a formal proof of the statement

$$\{\beta, \gamma\} \models \alpha \to \beta$$

which only uses the formulas from Λ and the Modus Ponens.

Homework 11.2
If $\{\alpha, \beta\}$ tautologically implies γ, is then the below derivation correct? Explain your answer.

1. $\{\alpha, \beta\} \vdash \alpha \to \beta \to \gamma$ (Axiom Group 1)
2. $\{\alpha, \beta\} \vdash \alpha$ (Copy)
3. $\{\alpha, \beta\} \vdash \beta \to \gamma$ (Modus Ponens)
4. $\{\alpha, \beta\} \vdash \beta$ (Copy)
5. $\{\alpha, \beta\} \vdash \gamma$ (Modus Ponens)

For the following exercises, P, Q are predicates and a, b, c are constants.

Homework 11.3
Make a formal proof for

$$\{\forall x\,[P(x) \to Q(c)], \forall x\,[\neg P(x) \to Q(c)]\} \vdash Q(c)$$

Homework 11.4
Make a formal proof for $\{\forall x\,[P(x)], \exists y\,[\neg P(y)]\} \vdash Q(z)$.

Homework 11.5
Make a formal proof for $\emptyset \vdash \forall x\,\forall y\,[P(x) \to Q(y)] \to P(a) \to Q(b)$.

Homework 11.6
Is the statement $\emptyset \vdash P(x) \to \forall y\,[P(y)]$ correct? Explain your answer.

Homework 11.7
Is the statement $\emptyset \vdash P(x) \to \forall y\,[P(x)]$ correct? Explain your answer.

Homework 11.8
Is the statement $\emptyset \vdash P(x) \rightarrow \exists y\, [P(y)]$ correct? Explain your answer.

Homework 11.9
Let $(G, \circ, f, e)$ be a structure and Γ contain the following axioms:

- $\forall x, y, z\, [(x \circ y) \circ z = x \circ (y \circ z)]$;
- $\forall x, y\, [x \circ y = y \circ x]$;
- $\forall x\, [x \circ e = x]$;
- $\forall x\, [x \circ f(x) = e]$;
- $\forall x, y, z\, [x \circ y = x \circ z \rightarrow y = z]$;

So $(G, \circ)$ is an Abelian group with neutral element e and inversion f. Prove informally the following results:

- $\forall v, w\, [f(v) = f(w) \rightarrow v = w]$;
- $\forall v, w\, [v \circ w = e \rightarrow f(v) = w]$;
- $\forall v, w\, [f(v \circ w) = f(w) \circ f(v)]$.

Homework 11.10
Consider all structures $(A, \circ)$ where A has two elements and satisfies the axioms

$$\forall x\, [x \circ x = x] \text{ and } \forall x\, \forall y\, [x \circ y = y \circ x].$$

Show that all these structures are isomorphic.

Homework 11.11
Assume that $(\mathbb{N}, +, <, 0, 1, P)$ is a structure where $\mathbb{N}$ is the set of natural numbers and $+, <, 0, 1$ have the usual meaning on $\mathbb{N}$. Let the powers of 2 be the set $\{1, 2, 4, 8, 16, \ldots\}$ and make a formula α such that $(\mathbb{N}, +, <, 0, 1, P) \models \alpha$ iff $\forall x\, [Px \leftrightarrow \text{x is a power of 2}]$.

Note that such a formula only implicitly defines the powers of 2 and not explicitly; therefore this formula α does *not say* that the powers are definable from addition and order in $\mathbb{N}$.

Homework 11.12
Make a formula α which says that $f : A \rightarrow A$ is a one-to-one function but not an onto function. Provide a model $(A, f, =)$ which satisfies α. Can A be finite?